Solutions to the Midterm Exam – Linear Algebra

Math 110, Spring 2018. Instructor: E. Frenkel

Problem 1. Let V be the subspace of $\mathbb{R}^3$ defined by the equation

$$a_1 + 2a_2 + 3a_3 = 0.$$

Find a basis of V and give a proof that it is indeed a basis.

Solution. We claim that

$$\beta = \left\{ \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} \right\}$$

is a basis (note: of course, it's just one of many possibilities). Observe that $\dim(V) = 2$. Indeed, $V = N(T)$, where $T : \mathbb{R}^3 \to \mathbb{R}$ is the linear transformation sending $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ to $a_1 + 2a_2 + 3a_3$. Since T sends $\begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix}$ to a, we obtain that $R(T) = \mathbb{R}$. Therefore, by Dimension Theorem, $\dim(V) = \dim(\mathbb{R}^3) - \dim(\mathbb{R}) = 2$. Since β consists of two elements, in order to prove that β is a basis of V, it is sufficient to prove that β is linearly independent. Two vectors are linearly independent if and only if they are not proportional to each other. Clearly, any scalar multiple of $\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$ has 0 as the third entry, whereas any scalar multiple of $\begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}$ has 0 as the second entry. Hence the two vectors in β are not proportional to each other.

Problem 2. Let T be the linear transformation $P_3(\mathbb{C}) \to P_3(\mathbb{C})$ given by the formula $T(p(t)) = p(t+1)$. Compute the matrix $[T]_\beta$, where β is the basis of monomials in t.

Solution. By definition, the ith column of $[T]_\beta$ is the coordinate vector of $T(t^i) = (t+1)^i$ with respect to β. Hence

$$[T]_\beta = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Problem 3. Prove that a vector space V over a field F is isomorphic to F^n (where n is a positive integer) if and only if $\dim V = n$.

Solution. Since this is an "if and only if" statement, we need to prove it in both directions. Let us first prove that if $\dim(V) = n$, then V is isomorhic to F^n. Choose a basis $\beta = \{x_1, \ldots, x_n\}$ of V. Then, since β spans V, every vector in V can be written in the form

$$v = \sum_{i=1}^{n} a_i x_i, \qquad a_i \in F.$$

Moreover, since β is linearly independent, the scalars a_i are uniquely defined for each v. Define a map $\phi_\beta : V \to F^n$ by the formula

$$\phi_\beta(v) = \begin{pmatrix} a_1 \\ \ldots \\ a_n \end{pmatrix}$$

In other words, $\phi_\beta(v) = [v]_\beta$. It is a linear transformation because $[v + w]_\beta = [v]_\beta + [w]_\beta$ and $[cv] = c[v]_\beta$ for any $c \in F$. Let us show that ϕ_β is invertible. Define the linear transformation $\psi_\beta : F^n \to V$ by the formula

$$\psi_\beta(\begin{pmatrix} a_1 \\ \ldots \\ a_n \end{pmatrix}) = \sum_{i=1}^{n} a_i x_i.$$

Then it follows from the definitions of ϕ_β and ψ_β that $\phi_\beta \circ \psi_\beta = \mathrm{I}_{F^n}$ and $\psi_\beta \circ \phi_\beta = \mathrm{I}_V$. Hence ϕ_β is an isomorphism.

Conversely, suppose that V and F^n are isomorphic. Then there is an isomorphism $\psi : F^n \to V$. Let $\{e_1, \ldots, e_n\}$ be the canonical basis of F^n and set $x_i = \psi(e_i)$. Then

$$\psi(\begin{pmatrix} a_1 \\ \ldots \\ a_n \end{pmatrix}) = \sum_{i=1}^{n} a_i x_i.$$

Since ψ is onto, the set $\beta = \{x_1, \ldots, x_n\}$ generates V. Let us show that β is a linearly independent subset of V. Suppose that

$$\sum_{i=1}^{n} a_i x_i = \mathbf{0}.$$

Since the left hand side is equal to $\psi(\begin{pmatrix} a_1 \\ \ldots \\ a_n \end{pmatrix})$, it follows that $\begin{pmatrix} a_1 \\ \ldots \\ a_n \end{pmatrix} \in N(\psi)$. Since ψ is one-to-one, it follows that $a_i = 0$ for all i, and so β is indeed a linearly independent subset of V. Therefore β is a basis of V, and $\dim(V) = n$.

Problem 4. Let V be a two-dimensional vector space over $\mathbb{R}$ and $T : V \to V$ a linear transformation. Suppose that $\beta = \{x_1, x_2\}$ and $\beta' = \{y_1, y_2\}$ are two bases in V such that

$$x_1 = y_1 - y_2, \qquad x_2 = 2y_1 - y_2.$$

Find $[T]_\beta$ if

$$[T]_{\beta'} = \begin{pmatrix} 3 & -2 \\ -1 & 2 \end{pmatrix}$$

Solution. According to the formula proved in the book,

$$[T]_\beta = C^{-1}[T]_{\beta'}C,$$

where

$$C = \left([x_1]_{\beta'}[x_2]_{\beta'}\right) = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$$

(note: the roles played here by β and β' are opposite to those in the book). Therefore we find

$$[T]_\beta = \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3 & -2 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 4 \end{pmatrix}$$

Problem 5. Let V be an n-dimensional vector space over a field F and T a linear transformation $V \to V$. Suppose that W is a k-dimensional subspace of V, which is T-invariant (that is, $\forall\, \mathbf{v} \in W$, we have $T(\mathbf{v}) \in W$). Prove that there is a basis β of V such that each of the first k columns of the matrix $[T]_\beta$ has the following property: its last $(n-k)$ entries are all equal to 0.

Solution. Choose a basis $\{x_1, \ldots, x_k\}$ of W. By Replacement Theorem, we can extend it to a basis $\beta = \{x_1, \ldots, x_n\}$ of V. In $[T]_\beta$, the ith row is $[T(x_i)]_\beta$. If $i = 1, \ldots, k$, then $T(x_i) \in W$ because W is T-invariant. But then $T(x_i), i = 1, \ldots, k$, is a linear combination of $x_1, \ldots, x_k$ only, which is equivalent to saying that in $[T(x_i)]_\beta$, where $i = 1, \ldots, k$, the last $(n-k)$ entries are all equal to 0. But these $[T(x_i)]_\beta, i = 1, \ldots, k$, are exactly the first k columns of the matrix $[T]_\beta$, so the desired statement is proved.

Problem 6. Define linear functionals $f_1 : P_1(\mathbb{R}) \to \mathbb{R}$ and $f_2 : P_1(\mathbb{R}) \to \mathbb{R}$ by the formulas

$$f_1(p(t)) = p(3), \qquad f_2(p(t)) = p(-1)$$

for all $p(t) \in P_1(\mathbb{R})$.

Find the basis of $P_1(\mathbb{R})$ for which $\{f_1, f_2\}$ is the dual basis.

Solution. By definition, the sought-after basis consists of the polynomials $p_1(t)$ and $p_2(t)$ such that $f_i(p_j(t)) = \delta_{i,j}$.

Writing $p_1(t) = a_1 + b_1 t$, we obtain

$$a_1 + 3b_1 = 1,$$

$$a_1 - b_2 = 0.$$

Solving this system, we get $a_1 = 1/4, b_1 = 1/4$.

Writing $p_2(t) = a_2 + b_2 t$, we obtain

$$a_2 + 3b_2 = 0,$$

$$a_2 - b_2 = 1.$$

Solving this system, we get $a_2 = 3/4, b_2 = -1/4$.

Hence the sought-after basis is $\{1/4 + 1/4t, 3/4 - 1/4t\}$.

Problem 7. Let V be an n-dimensional vector space over a field F and V^* the dual space. Given a subspace W of V, let W^0 be the subspace of V^* which consists of all

$$f : V \to F \qquad \text{such that} \qquad f(\mathbf{v}) = 0, \quad \forall\, \mathbf{v} \in W.$$

Prove that $\dim W^0 = n - \dim W$.

Solution. Choose a basis $\{x_1, \dots, x_k\}$ of W. By Replacement Theorem, we can extend it to a basis $\{x_1, \dots, x_n\}$ of V. By definition, $f \in W^0$ if and only if

$$f\left(\sum_{i=1}^{k} a_i x_i\right) = \mathbf{0}, \qquad \forall a_i \in F, \quad i = 1, \dots, k.$$

Since f is linear, this property is equivalent to

$$f(x_i) = \mathbf{0}, \qquad i = 1, \dots, k.$$

Let $\{f_1, \dots, f_n\}$ be the basis of V^* which is dual to $\{x_1, \dots, x_k\}$. Then any $f \in V^*$ can be written as

$$f = \sum_{i=1}^{n} b_i f_i, \qquad b_i \in F.$$

Since $f_i(x_j) = \delta_{i,j}$, we find that

$$f(x_i) = b_i.$$

Therefore $f \in W^0$ if and only if $b_i = 0$ for all $i = 1, \dots, k$. This means that for any $f \in W^0$, we have

$$f = \sum_{i=k+1}^{n} b_i f_i, \qquad b_i \in F.$$

Hence $\{f_{k+1}, \dots, f_n\}$ is a basis of W^0, and so $\dim(W^0) = n - k = \dim(V) - \dim(W)$.