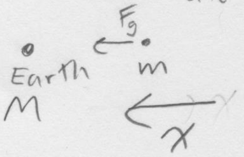
Mike Deweese MT2 Spring 2016 7A

1\. a.) N2L x: satellite



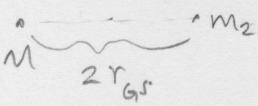
$$F_{net_x} = m a_x \Rightarrow \frac{G M \cancel{m}}{r_{GS}^{\cancel{2}}} = \cancel{m} \frac{v^2}{\cancel{r_{GS}}}$$ ← uniform circular motion

$$\frac{GM}{r_{GS}} = v^2 = \left(\frac{2\pi r_{GS}}{T}\right)^2 = \frac{(2\pi)^2 r_{GS}^2}{T^2}$$

$$r_{GS}^3 = \frac{GM}{(2\pi)^2} T^2$$

$$\boxed{r_{GS} = \left(\frac{GM}{4\pi^2} T^2\right)^{1/3}} \quad \text{— (1)}$$

b.) N2L x: satellite 2



$$\frac{GM\cancel{m_2}}{(2r_{GS})^2} = \cancel{m_2}\frac{v_2^2}{2r_{GS}} = \frac{\cancel{m_2}}{2r_{GS}}\left(\frac{2\pi\, 2r_{GS}}{T_2}\right)^2$$

$$\frac{GM}{4r_{GS}} = \frac{8\pi^2 r_{GS}^2}{T_2^2}$$

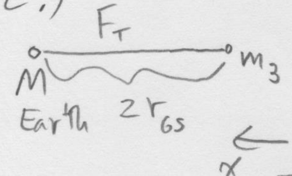
$$T_2 = \left(\frac{32\pi^2 r_{GS}^3}{GM}\right)^{1/2}$$

$$(1) \Rightarrow T \approx 1\,\text{day} = \left(\frac{4\pi^2}{GM} r_{GS}^3\right)^{1/2} \Rightarrow T_2 = T\left(\frac{32}{4}\right)^{1/2}$$

$$\boxed{T_2 = 2^{3/2} T}$$

(This also just follows from Kepler's 3rd law $\left(\frac{T_1}{T_2}\right)^2 = \left(\frac{S_1}{S_2}\right)^3$, where $r_i = S_i$ for circular orbits)

c.)



N2L x: object 3

$$\frac{GMm_3}{(2r_{GS})^2} + F_T = m_3 a_x = m_3\frac{v^2}{2r_{GS}} = m_3\frac{(2\pi\, 2r_{GS})^2}{2r_{GS}T^2}$$

$$\boxed{F_T = -\frac{GMm_3}{4r_{GS}^2} + \frac{m_3 8\pi^2 r_{GS}}{T^2}}$$ ← acceptable answer

$$(1) \Rightarrow F_T = m_3\left(\frac{-GM}{4\left(\frac{GM}{4\pi^2}T^2\right)^{2/3}} + \frac{8\pi^2}{T^2}\left(\frac{GM}{4\pi^2}T^2\right)^{1/3}\right)$$

$$F_T = m_3\left(-\left(\frac{G^3M^3 4^2\pi^4}{2T^4G^2M^2}\right)^{1/3} + \left(\frac{GMT^2 8^3\pi^6}{T^6 4\pi^2}\right)^{1/3}\right)$$

$$= m_3\left(-\left(\frac{GM\,8\pi^4}{T^4}\right)^{1/3} + \left(\frac{GM\,2^7\pi^4}{T^4}\right)^{1/3}\right) = \underbrace{m_3\left(2^{4/3}-1\right)\left(\frac{GM8\pi^4}{T^4}\right)^{1/3}}_{\text{in terms of } m_3, M, T, \& G}$$

1. d) use C.O.E.:

<u>immediately</u> after rope is cut: $v_i = \frac{2\pi 2 r_{GS}}{T} \overset{①}{=} \frac{4\pi}{T}\left(\frac{GMT^2}{4\pi^2}\right)^{1/3}$ ②

"escape velocity" for an object starting at a distance of r_{GS} (rather than the surface of the Earth) is given by Cons. of Energy assuming $v_f = 0$:

$$\left(\cancel{PE_{gf}}^{\,0} + \cancel{KE_f}^{\,0}\right)_f - \left(\frac{1}{2}m_3 v_{esc}^2 - \frac{G m_3 M}{r_{GS}}\right)_i = \cancel{W_{ext}}^{\,0}$$

← because W_g is accounted for by PE_g terms

$$\frac{1}{2}\cancel{m_3} v_{esc}^2 = \frac{G\cancel{m_3}M}{2r_{GS}}$$

$$v_{esc} = \left(\frac{GM}{r_{GS}}\right)^{1/2} \overset{①}{=} \left(\frac{GM}{\left(\frac{GM}{4\pi^2}T^2\right)^{1/3}}\right)^{1/2}$$

$$v_{esc} = \left(\frac{4\pi^2 G^3 M^3}{GMT^2}\right)^{1/6}$$

$$v_{esc} = \left(\frac{2^2\pi^2 G^2 M^2}{T^2}\right)^{1/6}$$

$$v_{esc} = 2^{1/3}\left(\frac{\pi G M}{T}\right)^{1/3} \quad ③$$

$$②: v_i = 2^{2-2/3}\left(\frac{\pi^3 GMT^2}{T^3\pi^2}\right)^{1/3}$$

$$= 2^{4/3}\left(\frac{\pi GM}{T}\right)^{1/3} \quad ④$$

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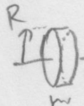
③ & ④: $2^{4/3} > 2^{1/3} \Rightarrow v_i > v_{esc}$ so the object <u>does</u> escape Earth's orbit and goes arbitrarily far away

Mike DeWeese solutions for MT2 spring 2016 7A

2. a.) solid cylinder w/ uniform density ρ, thickness d, and radius R:



$$I_{\substack{cm\\ \text{solid cylinder}}} = \int dm\, r_\perp^2 = \int_0^{2\pi} d\theta \int_0^R dr \int_0^d dz\; r\rho\, r^2 = 2\pi\rho d \int_0^R r^3 dr = 2\pi\rho d\frac{R^4}{4} = \frac{1}{2}\pi\rho d R^4$$

$$\underline{\tfrac{1}{2}M} = \int_0^{2\pi} d\theta \int_0^R dr \int_0^d dz\, r\rho = 2\pi d\rho \frac{R^2}{2} = \pi d\rho R^2 \Rightarrow I_{\substack{cm\\ \text{solid}\\ \text{cylinder}}} = \frac{1}{4}MR^2$$

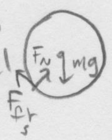
$$I_{\substack{cm\\ \text{hollow}\\ \text{cyl.}}} = \frac{1}{2}MR^2, \text{ since all mass is at } r=R$$

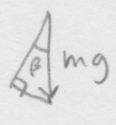
$$\text{so } I_{\substack{\text{wheel}\\ cm}} = I_{\substack{cm\\ \text{solid}\\ \text{cyl.}}} + I_{\substack{cm\\ \text{hollow}\\ \text{cyl.}}} = \left(\frac{1}{2}+\frac{1}{4}\right)MR^2 = \boxed{\frac{3}{4}MR^2} \quad (5)$$

b.) The normal force acts on the part of the wheel that is in contact with the ramp. The wheel rolls without slipping, so that part of the wheel is <u>not</u> moving, so $W_{\substack{F_N \text{ on}\\ \text{wheel}}} = \boxed{0}$ — (6)

c.)

FBD wheel





$$\text{N2L}_{\substack{\alpha\\ \text{wheel}}}: \cancel{\tau_N}^{\,0} + \cancel{\tau_g}^{\,0} + F_{fr_s}R = I_{cm}\alpha \quad (1)$$

accel. of c.m.

$$\text{N2L}_{\substack{x\\ \text{wheel}}}: \cancel{F_{N_x}}^{\,0} - F_{fr_s} + Mg\sin\beta = Ma_x \quad (2) \qquad g = +10\,\text{m/s}^2$$

$$a_x = R\alpha \quad (3)$$

$$(2)\ \&\ (3) \Rightarrow -F_{fr_s} + Mg\sin\beta = MR\alpha \quad (4)$$

$$(1)\ \&\ (4) \Rightarrow F_{fr_s}R = I_{cm}\frac{(-F_{fr_s} + Mg\sin\beta)}{MR}$$

$$F_{fr_s}(MR^2 + I_{cm}) = Mg\sin\beta\, I_{cm}$$

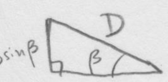
$$F_{fr_s} = \frac{Mg\sin\beta\, I_{cm}}{MR^2 + I_{cm}}$$

$$(5) \Rightarrow F_{fr_s} = \frac{Mg\sin\beta\ 3/4}{1 + 3/4} = \frac{Mg\sin\beta}{4/3 + 1} = \boxed{\frac{3}{7}Mg\sin\beta}$$

points <u>up</u> the hill

d.) C.O.E. $\left(\frac{1}{2}Mv_f^2 + \frac{1}{2}I_{cm}\omega_f^2 + Mgh_f\right) - \left(\cancel{KE_{tot_i}}^{\,0} + Mgh_i\right) = \cancel{W_{ext}}^{\,0}$ ← (6) & $W_{fr} = 0$

$$(5)\ \&\ v_f = R\omega_f \Rightarrow \frac{1}{2}MR^2\omega_f^2 + \frac{1}{2}\frac{3}{4}MR^2\omega_f^2 = Mg(h_i - h_f) = MgD\sin\beta$$

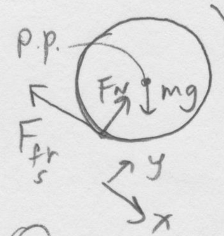


$$\frac{7}{8}\cancel{M}R^2\omega_f^2 = \cancel{M}gD\sin\beta \Rightarrow \boxed{\omega_f = \sqrt{\frac{8gD\sin\beta}{7R^2}}}$$

e.) (see next page)

2e.)

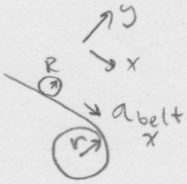
FBD wheel



$N2L_{wheel}\,\alpha: +F_{fr_s} R = I_{cm}\,\alpha_{wheel}$ ⑦

$N2L_{wheel}\,x: -F_{fr_s} + Mg\sin\beta = M a_x \to 0$ ⑧

⑦ & ⑧ $\Rightarrow Mg\sin\beta\, R = I_{cm}\,\alpha_{wheel} \underset{⑤}{=} \frac{3}{4} MR^2 \alpha_{wheel}$



$$\alpha_{wheel} = \frac{\cancel{M} g \sin\beta\, \cancel{R}}{\frac{3}{4}\cancel{M} R^{\cancel{2}}} = \frac{4 g \sin\beta}{3R} \quad ⑨$$

The accelleration of the belt is related to α of the wheel & α of the roller:

$$a_{belt_x} = +r\,\alpha_{roller} = -R\,\alpha_{wheel}$$

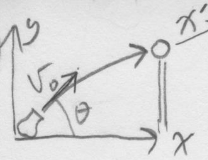
so $|\alpha_{roller}| = \frac{R}{r}\alpha_{wheel} \underset{⑨}{=} \frac{\cancel{R}}{r}\frac{4 g \sin\beta}{3\cancel{R}}$

$$\boxed{|\alpha_{roller}| = \frac{4 g \sin\beta}{3r}}$$

f.) ⑧ $\Rightarrow F_{fr_s} = Mg\sin\beta$

$W_{fr} = \vec{F}_{fr_s} \cdot \Delta\vec{r} = \boxed{MgD\sin\beta}$

(This is non-zero since the point of contact between the wheel and belt is moving in the direction of the force of friction on the wheel)

3a)


const. $\vec{a}$: $\quad t_{hit} = \frac{v_0 \sin\theta}{2g}$

$$v_x(t_{hit}) = v_{x0} + \cancelto{0}{a_x} t_{hit} = \boxed{v_0 \cos\theta}$$

$$v_y(t_{hit}) = v_{y0} + a_y t_{hit} = v_0 \sin\theta + (-\cancel{g}) \frac{v_0 \sin\theta}{2\cancel{g}}$$

$$= \boxed{\tfrac{1}{2} v_0 \sin\theta}$$

b.) immediately after collision, ball & Supremo velocities point in same direction means that this is a <u>1-D collision</u>.

bouncy rubber ball bounces of helmet $\Rightarrow$ <u>elastic</u> collision so K.E. is conserved as well as momentum.

Along the direction of motion, x':

C.O.E. $\quad \frac{1}{2} m v_{sfx'}^2 + \frac{1}{2} m v_{Bfx'}^2 = \frac{1}{2} m v_{six'}^2 + \frac{1}{2} m \cancelto{0}{v_{Bix'}^2}$

$$v_{sfx'}^2 + v_{Bfx'}^2 = v_{six'}^2 \quad (1)$$

C.O. $P_{x'}$: $\quad m v_{sfx'} + m v_{Bfx'} = m v_{six'} + m \cancelto{0}{v_{Bix'}}$

$$v_{sfx'} = v_{six'} - v_{Bfx'} \quad (2)$$

(2) in $(1) \Rightarrow (v_{six'} - v_{Bfx'})^2 + v_{Bfx'}^2 = v_{six'}^2$

$$\cancel{v_{six'}^2} - 2 v_{six'} v_{Bfx'} + v_{Bfx'}^2 + v_{Bfx'}^2 - \cancel{v_{six'}^2} = 0$$

$$\cancel{2} v_{six'} v_{Bfx'} = \cancel{2} v_{Bfx'}^2$$

$$\text{so } v_{Bfx'} = 0 \text{ or } v_{six'}$$

ball bounces off helmet $\Rightarrow v_{Bfx'} > v_{sfx'}$

also, we're told $\vec{v}_{Bf}$ is in same direction as $\vec{v}_{si}$, so $\vec{v}_{Bf} \neq 0$ since a vector with 0 magnitude has no direction

$$\text{so } |\vec{v}_{Bf}| = |\vec{v}_{si}| = \sqrt{v_0^2 \cos^2\theta + \tfrac{1}{4} v_0^2 \sin^2\theta}$$

$$= \boxed{v_0 \sqrt{1 - \tfrac{3}{4} \sin^2\theta}}$$

3.c) May be an inelastic collision (but not totally inelastic since ball and Supremo are not "stuck" together after collision) so kinetic E may not be cons., but $\vec{p}$ is always conserved:

in the horizontal direction: C.O. P_x: $mv_{Bfx} + mv_{sfx} = m\cancel{v_{Bix}}^{\,0} + mv_{six}$

$$v_{Bfx} + v_{sfx} = v_{six} \quad (3)$$

$$v_{Bfx} = 3v_{sfx} \quad (4)$$

$$(3) \,\&\, (4) \Rightarrow v_{Bfx} + \frac{1}{3}v_{Bfx} = v_{six}$$

$$|\vec{v}_{Bf}| \underset{\text{ball moves horizontally}}{=} v_{Bfx} = \frac{3}{4}v_{six} = \boxed{\frac{3}{4}v_0\cos\theta}$$

d.) impulse $= \Delta\vec{P}$ ball moves horizontally after collision, starts from rest.

$$\vec{J} = M\vec{v}_{Bf} - M\cancel{\vec{v}_{Bi}}^{\,0} = \boxed{M\frac{3}{4}v_0\cos\theta\ \hat{i}}$$

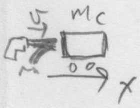
to the right in diagram

e.) holds onto ball after collision $\Rightarrow$ totally inelastic collision

$v_{fB} = v_{fs}$ C.O. $\vec{P}$: $\cancel{M}v_{six'} = \cancel{M}v_{sfx'} + \cancel{M}v_{Bfx'} = 2\cancel{M}v_{sfx'}$

$$|\vec{v}_{sf}| = v_{sfx'} = \frac{1}{2}v_{six'} = \boxed{\frac{v_0}{2}\sqrt{1-\frac{3}{4}\sin^2\theta}}$$

4.a)



$$\underset{\text{water}}{N2L}\ x: \quad F_{\substack{\text{cart} \\ \text{on} \\ \text{water} \\ x}} = \frac{dP_{water,x}}{dt} = \frac{dM_w}{dt}(v_{wxf} - v_{wxi})$$

$$= R(-v - v) \quad \leftarrow \text{elastic colision over very short time so } M_w \ll M_c$$

$$= -2Rv \quad -①$$

(only horizontal force on cart is from water)

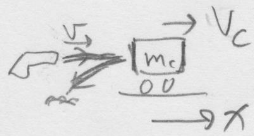
$$\underset{\text{cart}}{N2L}\ x: \quad F_{\substack{net \\ on\ c,x}} = m_c a_{cx} \quad -②$$

$$\underset{\text{cart \& water}}{N3L\ x}: \quad F_{\substack{c\ on\ w \\ x}} = -F_{\substack{w\ on\ c \\ x}} \quad -③$$

$$①, ② \& ③ \Rightarrow \quad a_{cx} = \frac{F_{w\ on\ c,x}}{m_c} = \frac{-F_{c\ on\ w,x}}{m_c} = \boxed{+\frac{2Rv}{m_c}}$$

($+x \Rightarrow$ to the right)

b.)



$$\underset{w}{N2L}\ x: \quad F_{c\ on\ w,x} = R(v_{wxf} - v_{wxi}) \quad -④$$

define x' to be horiz. axis of coordinate system moving with cart at this moment. For brief time period, $M_w \ll M_c$ so center of mass moves with the cart.

$$v_{x'} = v_x - v_c$$

$$v_{wix'} = v - v_c$$

$$v_{cix'} = v_{cfx'} = 0$$

so elastic col. $\Rightarrow$ $v_{wfx'} = -v_{wix'} = v + v_c$

$$v_{wfx} = v_{wfx} + v_c = -v + v_c + v_c$$

$$= -v + 2v_c$$

$$④ \Rightarrow \quad F_{c\ on\ w,x} = R(-v + 2v_c - v) = -2(v - v_c)R$$

$$④, ② \& ③ \Rightarrow \quad a_{cx} = \frac{-F_{c\ on\ w,x}}{m_c} = \boxed{\frac{+2(v - v_c)R}{m_c}}$$

c.) For short time immediately after gun starts shooting:

$$\underset{\substack{\text{water} \\ \text{leaving} \\ \text{gun}}}{N2L}\ x: \quad F_{\substack{\text{gun} \\ \text{on} \\ \text{water} \\ x}} = \frac{dP_{water,x}}{dt} = \frac{dM_w}{dt}(v_{wxf} - \cancelto{0}{v_{wxi}}) = Rv$$

$$N3L: \quad F_{\substack{g\ on\ w \\ x}} = -F_{\substack{w\ on\ g \\ x}}$$

$$\underset{\text{cart, gun, boy}}{N2L}: \quad F_{\substack{w\ on\ g \\ x}} = (m_B + m_w + m_c)\, a_{cx}$$

$$\boxed{a_{cx} = \frac{-Rv}{(m_B + m_w + m_c)}}$$

4d.) N2L$_x$ water leaving gun: $F_{g\,on\,w\,x} = \frac{dP_{w,x}}{dt} = \frac{dM_w}{dt}\left(v_{wxf} - v_{wxi}\right)$

$$= R\left(\left[v + v_{cx}\right] - \left[v_{cx}\right]\right)$$

$$= Rv$$

N3L$_x$: $F_{w\,on\,g\,x} = -Rv$

N2L$_x$ c&b: $F_{on\,g\,x} = \left(m_c + m_b + \cancel{m_{remaining\ H_2O}}^{\to 0}\right) a_{cx}$

$$\boxed{a_{cx} = \frac{-Rv}{(m_c + m_b)}}$$

e.) derive eq. for rocket propulsion at each moment in time: t_i: $dm \to u$ $\quad M \to v' \quad \to x$

C.O.P$_x$: t_f: $M+dm \longrightarrow v' + dv'$

$$\left((M+dm)(v'_x + dv'_x)\right)_f - \left(u_x dm + v'_x M\right)_i = dJ_x = \cancel{F}_{ext}^{\to 0}\, dt$$ no externally applied horizontal force.

$$\cancel{Mv'_x} + M dv'_x + v'_x dm + \cancel{dm \cdot dv'_x}^{\to 0} - u dm - \cancel{v'_x M} = 0$$ infinitesimal compared to other terms

$$M dv'_x = (u_x - v'_x)\, dm \quad \text{—}(1)$$

(I use a prime on "v'" to distinguish it from v in the problem description)

$v = u - v'$ is the velocity of the water w/respect to the gun

so (1) $\Rightarrow$
$$dv'_x = v\frac{dm}{m}$$

where M is the (time dependent) mass of the boy + cart + remaining H_2O and v' is the (time dependent) velocity of the cart

integrate both sides

$$\int_0^{v'_{fx}} dv'_x = \int_{m_i}^{m_f} v\frac{dm}{m}$$ (we want to find v'_f)

($M = m_b + m_c + m_w - Rt$ for $0 \le t \le \frac{m_w}{R}$; v is constant w/time)

so: $$v'_{fx} = v \ln(m)\Big|_{m_i}^{m_f} = v \ln\left(\frac{m_f}{m_i}\right)$$

$$\boxed{v'_{fx} = v \ln\left(\frac{m_b + m_c}{m_b + m_c + m_w}\right)}$$

(this sign convention assumes gun shoots water in $+x$ direction)