EECS 126 Probability and Random Processes University of California, Berkeley: Spring 2018
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Midterm 1

Last Name	First Name	SID

- You have 10 minutes to read the exam and 90 minutes to complete this exam.
- The maximum you can score is 120, but 100 points is considered perfect.
- The exam is not open book, but you are allowed to consult the cheat sheet that we provide. No calculators or phones.
- No form of collaboration between the students is allowed. If you are caught cheating, you may fail the course and face disciplinary consequences.
- Show all work to get any partial credit.
- Take into account the points that may be earned for each problem when splitting your time between the problems.

Problem	points earned	out of
Problem 1		30
Problem 2		20
Problem 3		10
Problem 4		20
Problem 5		20
Problem 6		20
Total		100 (+20)

Problem 1: Answer these questions briefly but clearly. [30]

(a) [**6**] Let $X \sim \mathcal{N}(0,1)$. Compute $\mathbb{E}[|X|]$.

One has

$$\mathbb{E}[|X|] = \int_{-\infty}^{\infty} |x| \frac{1}{\sqrt{2\pi}} \exp\Big(-\frac{x^2}{2}\Big)\, \mathrm{d}x = 2\frac{1}{\sqrt{2\pi}} \int_0^{\infty} x \exp\Big(-\frac{x^2}{2}\Big)\, \mathrm{d}x = \sqrt{\frac{2}{\pi}}.$$

(b) [**6**] Suppose that you are in a Second-Price Auction with n other bidders, and all other bidders draw their valuations uniformly and independently from the interval $[0,1]$. If your valuation is v, and everyone including you bids their own valuation, what is your expected profit from the auction? (Remember that if you don't win the auction, you don't pay anything.)

Let X_i be a random variable for the valuation of the ith bidder, and let $X = \max\limits_{1\le i\le n} X_i$. Then, your expected profit is

$$\mathbb{P}(X < v)\, \mathbb{E}[v - X \mid X < v] = v^n\Big(v - \int_0^v x \frac{nx^{n-1}}{v^n}\, \mathrm{d}x\Big) = v^n\Big(v - \frac{nv}{n+1}\Big) = \frac{1}{n+1} v^{n+1}.$$

(c) [**6**] Recall the soliton distribution we used for encoding messages in the fountain codes lab. In this setting, n data chunks were encoded into n packets, which are XORs of d many randomly selected data chunks, where d is drawn from a degree distribution $p(\cdot)$. Assuming a perfect channel, where the probability of a packet being erased is 0, **show that the probability that this scheme fails to decode the original message, $\mathbb{P}(\textbf{failure}) \geq \mathrm{e}^{-1}$ as $n \to \infty$.**

The Taylor series $\mathrm{e}^x = \lim_{n\to\infty}(1 + \frac{x}{n})^n$ may come in handy.

$$p(d) = \begin{cases} 0, & d \le 0 \text{ or } d > n \\ \frac{1}{n}, & d = 1 \\ \frac{1}{d(d-1)}, & 1 \le d \le n \end{cases}$$

The probability of failure is lower bounded by the probability that we are unable to start peeling, i.e. the probability that no degree 1 packets were originally sent.

$$\mathbb{P}(\text{no deg 1 packets}) = \mathbb{P}(\text{Packet 1 isn't degree 1})^n = \Big(1 - \frac{1}{n}\Big)^n \to \mathrm{e}^{-1}$$

(d) [**6**] Tom Brady just lost the Super Bowl, and he needs to validate that he is strong enough to keep playing football. He decides to throw a football as far as he can each day, and keep track of the farthest throw he has recorded so far; we will call this farthest throw his personal best. The distance that he throws the football is drawn from a continuous non-negative distribution and all throws are i.i.d. **What is the expected number of times, $\mathbb{E}[T]$, that his personal best changes over n throws?**

Assume that on day one, his personal best is distance 0 yards.

$$\mathbb{E}[T] = \sum_{i=1}^{n} \mathbb{E}[T_i] = \sum_{i=1}^{n} \frac{1}{i} \approx \ln n.$$

$T_i = 1$ if the ith throw goes farther than all the other throws he did previously. Because the ordering ranks of each of these throws is equally likely, the probability that the longest throw of these i occurred at the ith position is $1/i$.

(e) [6]

We consider a bag with red and blue balls. The number of red balls is Poisson distributed with parameter λ_{r}, and the number of blue balls is independently Poisson distributed with parameter λ_{b}. Conditioned on there being n (greater than 1) balls in the bag, what is the distribution of the number of blue balls? (Leave your answer in terms of n, λ_{r}, and λ_{b}.)

Let B be the number of blue balls and let R be the number of red balls. Then, for $b = 0, 1, \ldots, n$,

$$\begin{aligned}\mathbb{P}(B = b \mid B + R = n) &= \frac{\mathbb{P}(B = b, B + R = n)}{\mathbb{P}(B + R = n)} = \frac{\mathbb{P}(B = b, R = n - b)}{\mathbb{P}(B + R = n)} \\ &= \frac{\lambda_{\mathrm{b}}^{b} \mathrm{e}^{-\lambda_{\mathrm{b}}} \lambda_{\mathrm{r}}^{n-b} \mathrm{e}^{-\lambda_{\mathrm{r}}} / (b!(n-b)!)}{(\lambda_{\mathrm{b}} + \lambda_{\mathrm{r}})^{n} \mathrm{e}^{-(\lambda_{\mathrm{b}} + \lambda_{\mathrm{r}})} / n!} = \binom{n}{b} \Big(\frac{\lambda_{\mathrm{b}}}{\lambda_{\mathrm{b}} + \lambda_{\mathrm{r}}}\Big)^{b} \Big(\frac{\lambda_{\mathrm{r}}}{\lambda_{\mathrm{b}} + \lambda_{\mathrm{r}}}\Big)^{n-b}.\end{aligned}$$

Thus, $B \sim \text{Binomial}(n, p)$ where $p = \lambda_{\mathrm{b}}/(\lambda_{\mathrm{b}} + \lambda_{\mathrm{r}})$.

Problem 2: Bounds [20]

(a) [10] We have a random walk over the integers starting at zero. Each time we either move left or right with equal probability. Let S_n be a random variable which is equal to the integer that we lie on at time n. So $S_0 = 0$ and $S_n = Y_1 + \cdots + Y_n$, where each Y_i is either $+1$ or -1 with probability $1/2$, independently.

Show that

$$\mathbb{P}(|S_n| \geq t) \leq 2\mathrm{e}^{-\frac{t^2}{2n}}, \quad \text{for any } t \geq 0.$$

Hint: Recall that in homework 4 you showed that if $X_1, \ldots, X_n$ are independent Bernoulli(q), then

$$\mathbb{P}(|X_1 + \cdots + X_n - nq| \geq \epsilon) \leq 2\mathrm{e}^{-\frac{2\epsilon^2}{n}}, \quad \text{for any } \epsilon \geq 0.$$

Let $X_i = \frac{Y_i+1}{2}$ and observe that with this transform each $Y_i \sim$ Bernoulli$(1/2)$, independently. So using the result from the homework, with $\epsilon = t/2$ we have that

$$\begin{aligned}
\mathbb{P}\left(\left|\sum_{i=1}^{n} Y_i\right| \geq t\right) &= \mathbb{P}\left(\left|\sum_{i=1}^{n} 2X_i - n\right| \geq t\right) \\
&= \mathbb{P}\left(\left|\sum_{i=1}^{n} X_i - n\frac{1}{2}\right| \geq \frac{t}{2}\right) \\
&\leq 2\exp\left(-\frac{t^2}{2n}\right).
\end{aligned}$$

(b) [10] We have seen that the Markov bound can be quite loose. However, this need not always be true. For a given positive integer k, describe a random variable X that assumes only non-negative values such that: $\mathbb{P}(X \geq k\,\mathbb{E}(X)) = 1/k$.

Let $X = k$ with probability $1/k$ and $X = 0$ otherwise. Then $\mathbb{E}[X] = 1$ and

$$\mathbb{P}(X \geq k\,\mathbb{E}[X]) = \mathbb{P}(X \geq k) = \frac{1}{k}.$$

Problem 3: Entropy [10]

Recall the definitions of entropy and joint entropy for discrete random variables:

$$H(X) \triangleq \mathbb{E}[-\log_2 p_X(X)] = -\sum_x p_X(x)\log_2 p_X(x),$$
$$H(X,Y) \triangleq \mathbb{E}[-\log_2 p_{X,Y}(X,Y)] = -\sum_x\sum_y p_{X,Y}(x,y)\log_2 p_{X,Y}(x,y).$$

(a) **[5]** Let $U \sim \text{Uniform}\{1, 2, \ldots, n\}$. Give a closed form expression for $H(U)$.

$$H(U) = \mathbb{E}\Big[-\log_2 \frac{1}{n}\Big] = \log_2 n.$$

(b) **[5]** Assume that X, Y are independent. Express $H(X,Y)$ in terms of $H(X)$ and $H(Y)$.

$$\begin{aligned} H(X,Y) &= \mathbb{E}[-\log_2\{p_X(X)p_Y(Y)\}] \\ &= \mathbb{E}[-\log_2 p_X(X) - \log_2 p_Y(Y)] \\ &= \mathbb{E}[-\log_2 p_X(X)] + \mathbb{E}[-\log_2 p_Y(Y)] \\ &= H(X) + H(Y). \end{aligned}$$

Problem 4: Transformations of Random Variables [20]

Consider X, Y i.i.d. Uniform$[0,1]$. Let $Z = \ln \frac{X}{Y}$.

(a) **[5]** Show that $-\ln Y$ is exponentially distributed with parameter 1.

Let $W = -\ln Y$.

$$F_W(w) = \mathbb{P}(-\ln Y \le a) = \mathbb{P}(Y \ge \mathrm{e}^{-a}) = 1 - \mathrm{e}^{-a}$$

By pattern matching, we can see that $W \sim$ Exponential(1).

Alternate solution using the change of variables formula: Let $W = -\ln Y$. Then $f_W(w) = f_Y(h(w))|h'(w)|$, where $h(W) = Y$. Thus, $f_W(w) = \mathrm{e}^w f_Y(\mathrm{e}^w) = \mathrm{e}^w$, for $w > 0$, i.e., $W \sim$ Exponential(1). Therefore, $Z = Z_1 - Z_2$ where $Z_1, Z_2 \sim$ Exponential(1) are independent.

(b) **[5]** Find the moment generating function of Z, M_Z.

Observe that by independence,

$$M_Z(s) = M_{Z_1}(s) M_{Z_2}(-s) = \frac{1}{1+s}\frac{1}{1-s} = \frac{1}{1-s^2}, \qquad |s| < 1.$$

The MGF of Z can also be calculated from its density directly.

$$\begin{aligned} M_Z(s) = \mathbb{E}[\mathrm{e}^{sZ}] &= \int_{-\infty}^{\infty} \mathrm{e}^{sz}\frac{1}{2}\mathrm{e}^{-|z|}\,\mathrm{d}z = \int_{-\infty}^{0} \mathrm{e}^{sz}\frac{1}{2}\mathrm{e}^{z}\,\mathrm{d}z + \int_{0}^{\infty} \mathrm{e}^{sz}\frac{1}{2}\mathrm{e}^{-z}\,\mathrm{d}z \\ &= \frac{1}{2}\Big[\int_{-\infty}^{0} \mathrm{e}^{z(s+1)}\,\mathrm{d}z + \int_{0}^{\infty} \mathrm{e}^{z(s-1)}\,\mathrm{d}z\Big] = \frac{1}{2}\Big[\frac{\mathrm{e}^{z(s+1)}}{s+1}\Big|_{z=-\infty}^{0} - \frac{\mathrm{e}^{z(s-1)}}{s-1}\Big|_{z=0}^{\infty}\Big] \end{aligned}$$

We can simplify this to the following, however we must note that $|s| < 1$ for the integrals to converge.

$$M_Z(s) = \frac{1}{1-s^2}, \qquad |s| < 1$$

(c) **[5]** Find $\operatorname{var} Z$.

The easy way is to observe that by independence of Z_1 and Z_2, $\operatorname{var} Z = \operatorname{var} Z_1 + \operatorname{var} Z_2 = 2$.

Alternatively, $\mathbb{E}[Z] = 0$, by symmetry. Thus $\operatorname{var} Z = \mathbb{E}[Z^2]$.

$$\begin{aligned} M_Z'(s) &= \frac{2s}{(1-s^2)^2} \\ M_Z''(s) &= \frac{2(1-s^2)^2 - 2s(2(1-s^2))(-2s)}{(1-s^2)^4} \\ M''Z(s)\Big|_{s=0} &= 2 \end{aligned}$$

$$\operatorname{var} Z = 2$$

(d) [5] Find the PDF of Z, f_Z.

From here we can compute $\mathbb{P}(Z > z)$ (for $z > 0$) by conditioning:

$$\mathbb{P}(Z > z) = \mathbb{E}[\mathbb{P}(Z_1 - Z_2 > z \mid Z_2)] = \exp(-z)\,\mathbb{E}[\exp(-Z_2)] = \frac{1}{2}\exp(-z).$$

Thus, $f_Z(z) = \exp(-z)/2$ for $z > 0$, and by symmetry, we conclude that $f_Z(z) = \exp(-|z|)/2$.

Another method is to use partial fraction decomposition,

$$M_Z(s) = \frac{1}{1-s^2} = \frac{1}{2}\frac{1}{1-s} + \frac{1}{2}\frac{1}{1+s}$$

and then (by pattern matching) observe that $M_1(s) = 1/(1-s)$ is the MGF corresponding to density $f_1(x) = \mathrm{e}^{-x}\mathbb{1}\{x \geq 0\}$ and $M_2(s) = 1/(1+s)$ is the MGF corresponding to density $f_2(x) = \mathrm{e}^{x}\mathbb{1}\{x \leq 0\}$ so by inverting the MGF we get

$$f_Z(z) = \frac{1}{2}f_1(z) + \frac{1}{2}f_2(z) = \frac{1}{2}\mathrm{e}^{-z}\mathbb{1}\{z \geq 0\} + \frac{1}{2}\mathrm{e}^{z}\mathbb{1}\{z \leq 0\} = \frac{1}{2}\exp(-|z|).$$

Alternatively, we can convolve the two identically distributed random variables.

$$f_Z(z) = \int_{-\infty}^{\infty} f_W(w) f_W(w-z)\,\mathrm{d}w$$

Notice that $w \in (\ln 0, \ln 1) = (-\infty, 0)$. Similarly, $w - z \in (-\infty, z)$ or $w \in (-\infty, z)$. Thus the limits of the integral are $(-\infty, \min(z, 0))$. We can split it into two cases: $z < 0$ and $z \geq 0$.

$$z < 0 : f_Z(z) = \int_{-\infty}^{0} \mathrm{e}^{w}\mathrm{e}^{w-z}\,\mathrm{d}w = \mathrm{e}^{-z}\int_{-\infty}^{0} \mathrm{e}^{2w}\,\mathrm{d}w = \frac{1}{2}\mathrm{e}^{-z}$$
$$z \geq 0 : f_Z(z) = \int_{-\infty}^{z} \mathrm{e}^{w}\mathrm{e}^{w-z}\,\mathrm{d}w = \mathrm{e}^{-z}\int_{-\infty}^{z} \mathrm{e}^{2w}\,\mathrm{d}w = \frac{1}{2}\mathrm{e}^{z}$$

Combining the 2 parts, we have:

$$f_Z(z) = \frac{1}{2}\mathrm{e}^{-|z|}$$

Problem 5: Valentine's Day [20]

The joint density of (X, Y) is uniform on the shaded region in Figure 1. Mathematically, the shaded region consists of two half-circles (each of radius one) centered at $(-1, 0)$ and $(1, 0)$, along with a triangle in the lower half-plane.

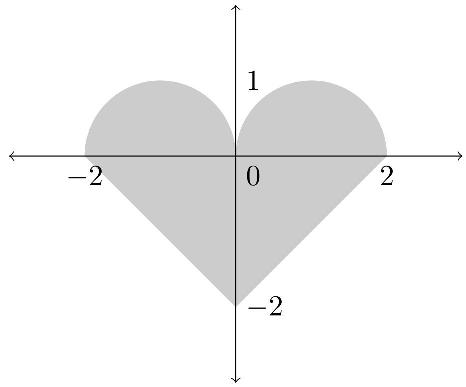

Figure 1: Joint density of (X, Y).

(a) **[7]** Find the value of the joint density in the shaded region.

The total area in the shaded region is $\pi + 4$, so the value of the joint density is $(\pi + 4)^{-1}$.

(b) **[8]** Find the marginal density of Y.

Since the joint density is uniform over the shaded region, the marginal density of Y at y is proportional to the width of the shaded region at vertical position y with constant of proportionality $(\pi + 4)^{-1}$.

$$f_Y(y) = \frac{2}{\pi + 4} \begin{cases} y + 2, & -2 < y < 0 \\ 2\sqrt{1 - y^2}, & 0 < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

(c) **[5]** Find $\operatorname{cov}(X, Y)$. (Don't handwave—explain your answer carefully.)

Note that $\mathbb{E}[X] = 0$ by symmetry. Also, $\mathbb{E}[XY] = \mathbb{E}[\mathbb{E}[XY \mid Y]] = \mathbb{E}[Y\,\mathbb{E}[X \mid Y]]$, but by symmetry, for each fixed $y \in [-2, 1]$ then $\mathbb{E}[X \mid Y = y] = 0$, so $\mathbb{E}[XY] = 0$. Hence, $\operatorname{cov}(X, Y) = 0$.

Problem 6: Waiting in Line [20]

You are first in line to be served by either of 2 servers, Alice and Bob, who are busy with their current customers. You will be served as soon as the first server finishes. The service times of Alice and Bob are independently and exponentially distributed with (positive) rates a and b respectively, i.e. Alice's distribution is Exponential(a), and Bob's is Exponential(b).

(a) **[5]** Find the probability that Alice will be your next server.

Let A denote Alice's service time and B denote Bob's service time. Then,

$$\mathbb{P}(A < B) = \mathbb{E}[\mathbb{P}(A < B \mid A)] = \mathbb{E}[\exp(-bA)] = \frac{a}{a+b}.$$

(b) **[5]** True or False? (you must prove your answer): The probability that you will be the last to be served (among you and the two current customers) is less than a half if and only if a is not equal to b.

By the Memoryless Property, if Alice finishes first, the probability that you are the last customer to be served is $b/(a+b)$ by the same reasoning in (a) (Bob's service time, which is now a fresh exponential, must beat Alice's); if Bob finishes first, the probability that you are the last customer to be served is $a/(a+b)$. Thus,

$$\mathbb{P}(\text{you are last to be served}) = \frac{a}{a+b}\frac{b}{a+b} + \frac{b}{a+b}\frac{a}{a+b} = \frac{2ab}{(a+b)^2}.$$

Then, $2ab/(a+b)^2 < 1/2$ if and only if $4ab < a^2 + b^2 + 2ab$, i.e., $2ab < a^2 + b^2$, but equality only holds when $a = b$, so the answer is True.

Proof: $0 \le (a-b)^2 = a^2 + b^2 - 2ab$ with equality if and only if $a = b$, so $2ab \le a^2 + b^2$ with equality if and only if $a = b$.

(c) **[10]** What is the distribution of your wait time? Your answer should not include integrals.

We consider the waiting time $W = \min(A, B)$.

$$\mathbb{P}(W > w) = \mathbb{P}\big(\min(A, B) > w\big) = \mathbb{P}(A > w) \cdot \mathbb{P}(B > w) = \mathrm{e}^{-wa} \cdot \mathrm{e}^{-wb} = \mathrm{e}^{-w(a+b)}$$

Thus we can see that $W \sim \text{Exponential}(a + b)$.