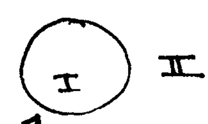
① Packard Midterm 2, Spring 2008

Problem 1 Solutions Michelle Yong

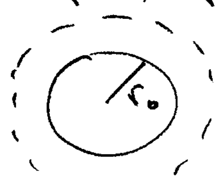
a) Because the charge distribution is spherically symmetric, the $\vec{E}$ field is sufficiently easy to find using Gauss's law, so we can find the potential from $V\left(\frac{r_0}{2}\right) - \cancel{V(\infty)}^{\,0} = -\int_{\infty}^{r_0/2} \vec{E}\cdot d\vec{l}$

$$= -\int_{\infty}^{r_0} \vec{E}_{II}\cdot d\vec{l} - \int_{r_0}^{\frac{r_0}{2}} \vec{E}_{I}\cdot d\vec{l}$$ where we need to split up the integral b/c the form of $\vec{E}$ is different inside & outside the sphere.

Let $r > r_0$ be region II & $r<r_0$, I.



$\boxed{\text{II}:}$



Gaussian surface → sphere of radius $r > r_0$

$\oint \vec{E}\cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0}$; by spherical symmetry, $\vec{E}$ only depends on r & points in $\hat{r}$ direction

$d\vec{a} = r^2 \sin\theta \, d\theta \, d\varphi \; \hat{r}$ (for a spherical surface)

$\vec{E}\cdot d\vec{a} = |\vec{E}|\, r^2 \sin\theta \, d\theta \, d\varphi$; $0<\theta<\pi$, $0<\varphi<2\pi$

$\oint \vec{E}\cdot d\vec{a} \Rightarrow |\vec{E}|\, 4\pi r^2$

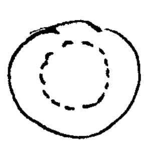
$Q_{enc} = \int_0^{r_0} \rho \, d\tau = \int_0^{r_0} \rho r^2 \sin\theta \, d\theta \, d\varphi \, dr$, $\rho = \alpha r$

$$Q_{enc} = \alpha 4\pi \int_0^{r_0} r^3 dr = \frac{\alpha \cancel{4}\pi r_0^4}{\cancel{4}} = \alpha \pi r_0^4$$

$$|\vec{E}|\, 4\pi r^2 = \frac{\alpha\pi r_0^4}{\epsilon_0} \Rightarrow \boxed{\vec{E}_{II}(r) = \frac{\alpha \cancel{\pi} r_0^4}{4\cancel{\pi}\epsilon_0 r^2}\hat{r}}$$

like a point charge of $\alpha\pi r_0^4$ @ origin

$\boxed{\text{I}:}$



Gaussian surface → sphere of radius $r < r_0$

Packard Midterm 2, Spring 2008

Michelle Yong

Problem 1 (CONT.) Solutions

$\boxed{\text{I :}}$ $Q_{enc} = \int_0^r \rho \, d\tau \rightarrow \alpha \pi r^4$

$$|\vec{E}| 4\pi r^2 = \frac{\alpha \pi r^4}{\epsilon_0} \implies \boxed{\vec{E}_I(\vec{r}) = \frac{\alpha r^2}{4\epsilon_0} \hat{r}}$$

$$V\left(\frac{r_0}{2}\right) = -\int_\infty^{r_0} \vec{E}_{II} \cdot d\vec{l} - \int_{r_0}^{r} \vec{E}_I \cdot d\vec{l}$$

$\left[d\vec{l} \rightarrow d\vec{r} = (dr)\hat{r} \right]$ $\rightarrow -\int_\infty^{r_0} \frac{\alpha r_0^4}{4\epsilon_0 r^2} dr - \int_{r_0}^{\frac{r_0}{2}} \frac{\alpha r^2}{4\epsilon_0} dr$

$$V\left(\frac{r_0}{2}\right) = \frac{\alpha r_0^4}{4\epsilon_0 r_0} - \frac{\alpha}{(4\epsilon_0)3}\left(\frac{r_0^3}{8} - r_0^3\right) = \frac{\alpha r_0^3}{\epsilon_0} \frac{31}{96}$$

$$\boxed{V\left(\frac{r_0}{2}\right) = \frac{\alpha r_0^3}{\epsilon_0} \frac{31}{96}}$$

b) use energy density of $\vec{E}$ field & integrate over volume

$\implies$ total energy $U = \int_{r_0}^{\infty} u_e \, d\tau$

energy density $u_e = \frac{1}{2}\epsilon_0 E_{II}^2 = \frac{1}{2}\epsilon_0 \left(\frac{\alpha r_0^4}{4\epsilon_0 r^2}\right)^2 = \frac{\alpha^2 r_0^8}{32\epsilon_0 r^4}$

$d\tau = r^2 \sin\theta \, d\theta \, d\varphi \, dr \Rightarrow 4\pi r^2 dr$

because u_e is spherically symmetric (doesn't depend on θ, φ)

$$U = \int_{r_0}^{\infty} \frac{\alpha^2 r_0^8}{32\epsilon_0 r^4} 4\pi r^2 dr = \frac{\alpha^2 r_0^8 \pi}{8\epsilon_0} \int_{r_0}^{\infty} \frac{dr}{r^2}$$

$$\boxed{U = \frac{\alpha^2 r_0^7 \pi}{8\epsilon_0}}$$

* since $\rho = \alpha r$, α has units $\left[\frac{\text{charge}}{(\text{length})^4}\right]$ because ρ has units $\left[\frac{\text{charge}}{\text{volume}}\right]$ then (thankfully) this answer has correct units!

Packard Midterm 2 Spring 2008
Physics 7B Solutions

Michelle Yong

Problem 1

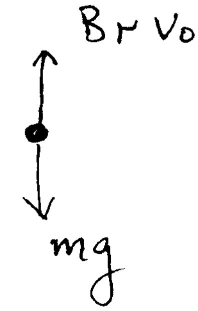
b) Alternative method: One might imagine a charged shell at infinity of charge $-Q = -\alpha\pi r_0^4$, so that we have a ~~odd~~ monstrous spherical capacitor. We expect that the infinitely huge shell has some potential ~~but~~ ~~then~~ which might cause problems, but then we remember that only the difference in potential between the two objects matters. The electric field ~~E~~ E_{II} is not affected because the huge shell is unseen @ infinity. The potential between the two is just $V(r_0) = \frac{Q_{tot}}{4\pi\epsilon_0 r_0} = \frac{\alpha\pi r_0^4}{4\pi\epsilon_0 r_0}$. The energy stored in the electric field is the same as that in our imaginary capacitor, which is $\frac{1}{2}QV$

$$U = \frac{1}{2}(Q_{tot})V(r_0) = \frac{1}{2}(\alpha\pi r_0^4)\frac{\alpha r_0^3}{4\epsilon_0} = \frac{1}{8}\frac{\alpha^2\pi r_0^7}{\epsilon_0}$$

$$\boxed{U = \frac{\alpha^2\pi r_0^7}{8\epsilon_0}}$$

Midterm 2 Packard

\#2 Particle falls at <u>constant</u> speed v_0

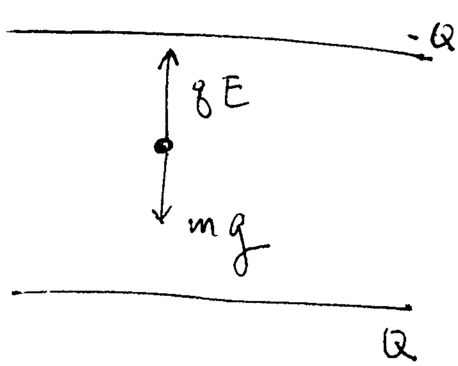


$\Rightarrow$ Net force $= 0$; $B r v_0 = \underbrace{\frac{4\pi}{3} r^3 \rho}_{\text{mass } m} g$

$$\Rightarrow \boxed{r = \left(\frac{3}{4\pi} \frac{B v_0}{\rho g} \right)^{1/2}}$$

Particle comes to rest when

$$qE = mg$$



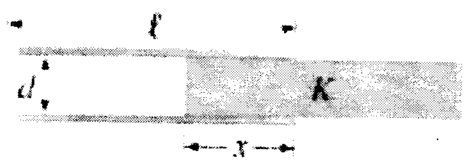
$$E = 2 \underbrace{\left(\frac{\sigma}{2\epsilon_0} \right)}_{E \text{ for 1 plate}} = \frac{Q}{A\epsilon_0}$$

$$\boxed{q = \frac{mg}{E} = \left(\frac{4\pi}{3} r^3 \rho g \right) \left(\frac{A\epsilon_0}{Q} \right)}.$$

3. A slab of width d and dielectric constant K is inserted a distance x into the space between the square, parallel plates (with side length ℓ and plate separateion d) of a capacitor as shown below. Determine, as a function of x:

a) The capacitance, $C(x)$.

b) The energy stored by the capacitor, $U(x)$, if the potential difference of the capacitor is V_0.



Treat them as parallel capacitors;

a) $C_{Total}(x) = C_{vacuum}(x) + C_{dielectric}(x)$

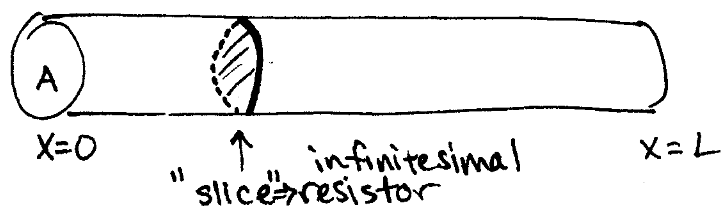
$$= \frac{\ell(\ell-x)\epsilon_0}{d} + \frac{\ell x \epsilon_0 K}{d}$$

$$= \boxed{\frac{\ell\epsilon_0}{d}\left(\ell+(K-1)x\right)}$$

b) $U(x) = \frac{1}{2}C(x)V_0^2 = \boxed{\frac{\ell\epsilon_0 V_0^2}{2d}\left(\ell+(K-1)x\right)}$

① Packard Midterm 2 Solutions, Spring 2008 for $\boxed{\text{Problem 4}}$ by Michelle Yong

a) In general, $R = \frac{\rho l}{A}$, but the resistivity changes along the length of the wire, so we need to treat the wire as made up of many infinitesimal slices, each of which has a resistance $dR = \frac{\rho\, dx}{A}$. Since resistors in series add like $R_{eff} = \sum_i R_i$, we will use an integral to sum over the infinitesimal resistors.



$$R_{eff} = \int_0^L \frac{\rho\, dx}{A} = \int_0^L \frac{\rho_0 e^{-x/L}}{A} dx = \frac{\rho_0}{A}(-L)\left(e^{-x/L}\right)\Big|_0^L$$

$$\boxed{R_{eff} = \frac{\rho_0 L}{A}\left(1 - e^{-1}\right) = \text{resistance across ends of wire}}$$

b) Using current density $\vec{J} = \frac{\vec{I}}{A} \cancel{= \sigma \vec{E} = \frac{\vec{E}}{\rho}}$ ✓ Ohm's law (a variant)

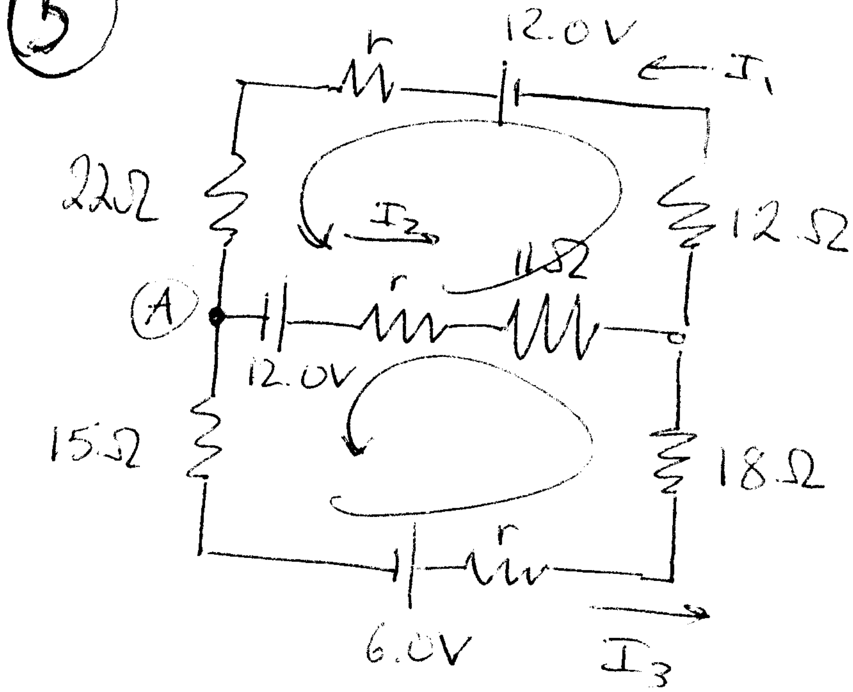
$\vec{J} = \sigma \vec{E} = \frac{\vec{E}}{\rho}$, we get $\boxed{\vec{E}(x) = \frac{\rho_0 I}{A} e^{-x/L}\,\hat{x}}$

$\boxed{\text{OR}}$ We can say each slice dR has the same current I through it due to an infinitesimal voltage drop $-dV$. then & since normally $I = \frac{V}{R}$, we have $I = \frac{-dV}{dR} = -dV\left(\frac{A}{\rho\, dx}\right) = \frac{A}{\rho}\left(\frac{-dV}{dx}\right) = \frac{A}{\rho} E$

then $\vec{E}(x) = \frac{\rho_0 I}{A} e^{-x/L}\,\hat{x}$ as above.

5

Peter



a. To determine the currents I_1, I_2, I_3, we need three independent equations.
Let's use use the junction rule at (A) and the two loops drawn.

Junction rule at (A):

$$\boxed{I_1 = I_2 + I_3}$$

bottom loop (starting at (A)):

$$-I_3 \cdot 15\,\Omega + 6\,V - I_3 \cdot 1\,\Omega - I_3\,18\,\Omega + I_2 \cdot 11\,\Omega + I_2 \cdot 1\,\Omega - 12\,V = 0$$

$$\boxed{12 I_2 - 34 I_3 = 6\cancel{V}A}$$

top loop:

$$12\,V - I_2 \cdot 1\,\Omega - I_2 \cdot 11\,\Omega - I_1\,12\,\Omega + 12\,V - I_1 \cdot 1\,\Omega - I_1 \cdot 22\,\Omega = 0$$

$$\boxed{35 I_1 + 12 I_2 = 24\cancel{V}A}$$

And here's how you solve those 3 equations with your graphing calculator:

$$\begin{bmatrix} -1 & +1 & +1 \\ 0 & 12 & -34 \\ 35 & 12 & 0 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 6A \\ 24A \end{bmatrix}$$

$$\underline{\underline{M}} \quad \underline{I} = \underline{b}$$

$$\underline{I} = \underline{\underline{M}}^{-1} \underline{b} = \begin{bmatrix} 0.511\text{ A} \\ 0.508\text{ A} \\ 0.00297\text{ A} \end{bmatrix}$$

So $\boxed{\begin{aligned} I_1 &= 0.511\text{ A} \\ I_2 &= 0.508\text{ A} \\ I_3 &= 0.00297\text{ A} \end{aligned}}$

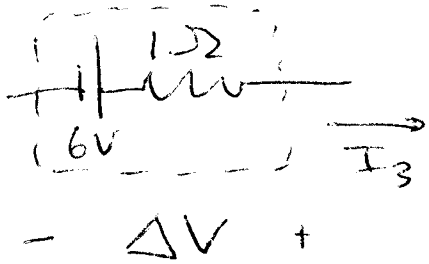
If you didn't have a calculator, the exact answers are:

$$I_1 = \frac{24}{47}\text{A} + \frac{12}{47}\frac{6}{2018}\text{A}$$

$$I_2 = \frac{24}{47}\text{A} - \frac{35}{47} \cdot \frac{6}{2018}\text{A}$$

$$I_3 = \frac{6}{2018}\text{ A}$$

b. To find the terminal voltage of the 6.0 V battery, we find the ΔV across



$$\Delta V = +6V - I_3 \cdot 1\Omega = 6V - \frac{6}{2018} A \cdot 1\Omega$$

$$\boxed{\Delta V = 5.997\ V}$$