University of California, Berkeley
Department of Mechanical Engineering
ME 104, Fall 2016

Final Examination (14 December 2016)

- Please write your name on the booklet every sheet.
- Start each problem on a new sheet.
- Write on <u>one</u> side of the page only.
- Number the sheets.

Problem 1

Consider a particle B moving along a space curve $\mathcal{C}$. Let s be the arclength of $\mathcal{C}$. The position vector of B at time t is given by

$$\mathbf{r} = \chi(B, t) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}. \tag{1}$$

Let $\{\mathbf{e}_t, \mathbf{e}_n, \mathbf{e}_b = \mathbf{e}_t \times \mathbf{e}_n\}$ be the Serret-Frenet basis and recall that

$$\frac{d\mathbf{e}_t}{ds} = \kappa\,\mathbf{e}_n, \tag{2}$$

where $\kappa \geq 0$ is the curvature of $\mathcal{C}$.

(a) Show that the velocity and acceleration of B may be expressed as

$$\mathbf{v} = \dot{s}\,\mathbf{e}_t, \quad \mathbf{a} = \ddot{s}\,\mathbf{e}_t + \kappa\dot{s}^2\,\mathbf{e}_n. \tag{3}$$

(b) Suppose that during an interval of time, the coordinates of a particle moving along a channel in a horizontal plane are specified in meters by

$$x = 0.3\,t, \quad y = 0.5\,t^2. \tag{4}$$

Calculate the velocity and acceleration of the particle and evaluate them at $t = 0.2\,\mathrm{s}$.

(c) Calculate the unit tangent vector $\mathbf{e}_t$ and unit normal vector $\mathbf{e}_n$ at $t = 0.2\,\mathrm{s}$.

(d) Calculate the tangential and normal components of acceleration at $t = 0.2\,\mathrm{s}$.

(e) Calculate the radius of curvature, $\rho = \dfrac{1}{\kappa}$, for $t = 0.2\,\mathrm{s}$.

Problem 2

Consider a rigid plate which is rotating with constant angular velocity $\boldsymbol{\omega} = 2.5\,\mathbf{k}$ rad/s about a vertical axle OZ. Place a corotational basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 = \mathbf{k}\}$ on the plate. Suppose that a particle ~~B~~ A is moving across the plate and that its position vector is given by

$$\mathbf{r} = x_1\,\mathbf{e}_1 + x_2\,\mathbf{e}_2. \tag{5}$$

(a) Show that the velocity of ~~B~~ A can be expressed as

$$\mathbf{v} = \mathring{\mathbf{r}} + \boldsymbol{\omega} \times \mathbf{r}, \tag{6}$$

where

$$\mathring{\mathbf{r}} = \dot{x}_1\,\mathbf{e}_1 + \dot{x}_2\,\mathbf{e}_2 \tag{7}$$

is the corotational rate of $\mathbf{r}$.

(b) Show that the acceleration of A can be expressed as

$$\dot{\mathbf{v}} = \overset{\circ\circ}{\mathbf{r}} + 2\boldsymbol{\omega} \times \overset{\circ}{\mathbf{r}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}), \tag{8}$$

where $\overset{\circ\circ}{\mathbf{r}}$ is the corotational rate of $\overset{\circ}{\mathbf{r}}$.

(c) What is the velocity and acceleration of the particle A' that belongs to the plate and with which A is coincident at the instant t?

(d) If x_1 and x_2 are specified in meters by

$$x_1 = 0.3\,t, \qquad x_2 = 0.2\,t^2, \tag{9}$$

calculate the magnitude of the velocity of B at $t = 5$ s.

Problem 3

A body $\mathcal{B}_1$ of mass $2m$ kg is sliding along a smooth horizontal rod with velocity v m/s. It collides with a stationary body $\mathcal{B}_2$ of mass m kg and becomes attached to it. The pair $\mathcal{B}_1$ and $\mathcal{B}_2$ subsequently collide with another stationary body $\mathcal{B}_3$ of mass m kg and become attached to it.

(a) Use Newton's second and third laws to prove that the linear momentum of the combined body $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3$ is conserved.

(b) Calculate the velocity of $\mathcal{B}$ after the second impact.

(c) Calculate the fractional loss, $\dfrac{T - T'}{T}$, of kinetic energy.

(d) If, after some time, all three bodies become separated again, write an equation that their velocities v_1', v_2', v_3' must satisfy. Without any further information, can we solve for v_1', v_2' and v_3' (Explain.)?

Problem 4

Let OZ be a rigid massless vertical axle. Let three particles, each of mass m kg, be attached to the axle by massless rigid rods, each of length ℓ m, and lying along the X and Y axes, as indicated in Fig.1.

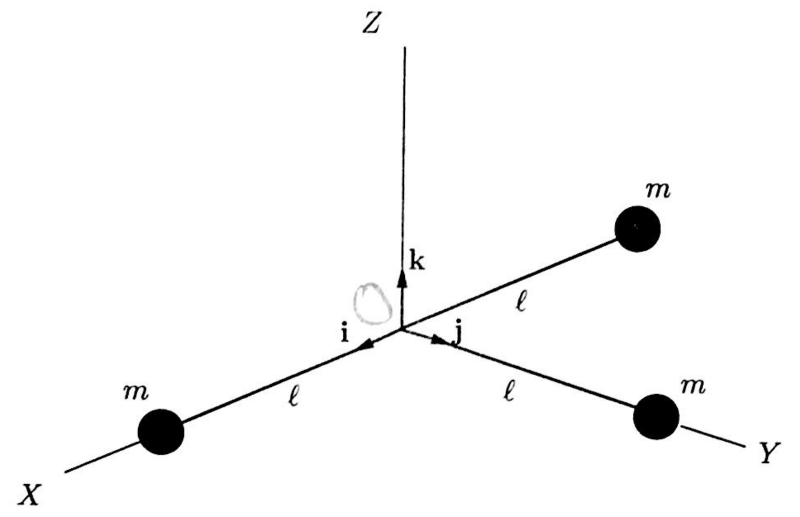

Figure 1: X,Y,Z are fixed axes in a Newtonian space

Suppose that the particles rotate rigidly about OZ with angular velocity $\boldsymbol{\omega} = 2t\,\mathbf{k}$ rad/s. Attach a corotational basis $\{\mathbf{e}_1(t), \mathbf{e}_2(t), \mathbf{e}_3(t)\}$ to the body, with $\mathbf{e}_1(0) = \mathbf{i}$, $\mathbf{e}_2(0) = \mathbf{j}$, and $\mathbf{e}_3 = \mathbf{k}$ for all t.

(a) Calculate the linear momentum of the body $\mathscr{B}$ consisting of the 3 particles. Calculate the resultant force that must act on $\mathscr{B}$.

(b) The inertia tensor of $\mathscr{B}$ about O is defined by

$$\mathbf{I}^O = \sum_{K=1}^{3} m_K \left[(\mathbf{r}_K \cdot \mathbf{r}_K)\mathbf{I} - \mathbf{r}_K \otimes \mathbf{r}_K \right], \tag{10}$$

where $\mathbf{I}$ is the identity tensor and $\otimes$ denotes the tensor product $[(\mathbf{a} \otimes \mathbf{b})\mathbf{u} = \mathbf{a}(\mathbf{b} \cdot \mathbf{u})]$. Also, the angular momentum of $\mathscr{B}$ about O is given by

$$\mathbf{H}^O = \mathbf{I}^O \boldsymbol{\omega}. \tag{11}$$

Evaluate $\mathbf{I}^O$ and $\mathbf{H}^O$.

(c) Apply Euler's second law to calculate the resultant torque that must act on $\mathscr{B}$.

(d) What are the eigenvectors (or principal directions) and the eigenvalues (or principal moments of inertia) of the inertia tensor $\mathbf{I}^O$ of $\mathscr{B}$?

Problem 5

Consider a 3-dimensional rigid body $\mathscr{B}$ which is rotating about an axle lying along a fixed axis OZ. Let the unit vector $\mathbf{k}$ point along OZ, and let $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ be a fixed orthonormal basis in a Newtonian frame of reference. Define a right-handed orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, which is fixed to the body (i.e., it is corotational), by

$$\begin{aligned} \mathbf{e}_1 &= \mathbf{Q}(\mathbf{k}, \theta)\,\mathbf{i} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}, \\ \mathbf{e}_2 &= \mathbf{Q}(\mathbf{k}, \theta)\,\mathbf{j} = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}, \\ \mathbf{e}_3 &= \mathbf{Q}(\mathbf{k}, \theta)\,\mathbf{k} = \mathbf{k}, \end{aligned} \tag{12}$$

where $\mathbf{Q}(\mathbf{k}, \theta)$ is the rotation tensor for B, and is given by

$$\mathbf{Q}(\mathbf{k}, \theta) = \cos\theta\,(\mathbf{i}\otimes\mathbf{i} + \mathbf{j}\otimes\mathbf{j}) - \sin\theta\,(\mathbf{i}\otimes\mathbf{j} - \mathbf{j}\otimes\mathbf{i}) + \mathbf{k}\otimes\mathbf{k} = \mathbf{e}_1\otimes\mathbf{i} + \mathbf{e}_2\otimes\mathbf{j} + \mathbf{e}_3\otimes\mathbf{k}. \tag{13}$$

(a) Calculate the value of the rotation tensor $\mathbf{Q}(\mathbf{k}, 40^\circ)$ and write out the matrix of $\mathbf{Q}(\mathbf{k}, 40^\circ)$, using the vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ to form a basis.

(b) If a particle $A \in \mathscr{B}$ is located at

$$\mathbf{R}_A = 0.2\,\mathbf{i} - 0.5\,\mathbf{j} + 0.3\,\mathbf{k} \quad \text{m} \tag{14}$$

in the initial configuration of $\mathscr{B}$ ($\theta = 0$), find its new location

$$\mathbf{r}_A = \mathbf{Q}(\mathbf{k}, 40^\circ)\,\mathbf{R}_A. \tag{15}$$

Problem 6

Consider a rigid body $\mathscr{B}$ of mass m rotating about a fixed axis OZ in a Newtonian space, with angular velocity $\boldsymbol{\omega} = \omega\,\mathbf{k} = \dot{\theta}\,\mathbf{k}$. Place a corotational basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 = \mathbf{k}\}$ on $\mathscr{B}$, with $\mathbf{e}_1 = \mathbf{i}$, $\mathbf{e}_2 = \mathbf{j}$ at time $t = 0$. The inertia tensor about O at time t may be expressed as

$$\mathbf{I}^O = I^O_{XX}\,\mathbf{e}_1\otimes\mathbf{e}_1 + I^O_{XY}\,\mathbf{e}_1\otimes\mathbf{e}_2 + \dots + I^O_{ZZ}\,\mathbf{e}_3\otimes\mathbf{e}_3. \tag{16}$$

(a) Recall that the angular momentum of $\mathscr{B}$ about O may be expressed as

$$\mathbf{H}^O = \mathbf{I}^O\,\boldsymbol{\omega}. \tag{17}$$

Resolve Eq. (17) on the corotational basis.

(b) If

$$\mathbf{M}^O = M^O_1\,\mathbf{e}_1 + M^O_2\,\mathbf{e}_2 + M^O_3\,\mathbf{e}_3 \tag{18}$$

is the resultant torque acting on $\mathscr{B}$, prove that

$$
\begin{aligned}
M_1^O &= I_{XZ}^O\,\dot{\omega} - I_{YZ}^O\,\omega^2, \\
M_2^O &= I_{YZ}^O\,\dot{\omega} + I_{XZ}^O\,\omega^2, \qquad (19) \\
M_3^O &= I_{ZZ}^O\,\dot{\omega}.
\end{aligned}
$$

(c) Consider an unbalanced rotor for which

$$I_{XZ}^O = 2\ \mathrm{kg}\cdot\mathrm{m}^2, \qquad I_{YZ}^O = 4\ \mathrm{kg}\cdot\mathrm{m}^2, \qquad I_{ZZ}^O = 150\ \mathrm{kg}\cdot\mathrm{m}^2, \tag{20}$$

and $\boldsymbol{\omega} = \omega(t)\,\mathbf{e}_3$, $\omega(0) = 0$. Suppose that during an interval of time $0 \le t \le 25$ s, a torque

$$M_3^O = 30\ \mathrm{N}\cdot\mathrm{m} \tag{21}$$

is being supplied by a motor. Calculate $\omega(t)$, $H_1^O(t)$, $H_2^O(t)$, $H_3^O(t)$, $M_1^O(t)$, and $M_2^O(t)$, and evaluate these at $t = 25$ s.

Problem 7

Consider a compound pendulum that consists of massless rigid rod to which is attached a rigid ball of mass m kg and radius r m (see Fig.2). The distance from the pivot O to the mass center C of the ball is ℓ m.

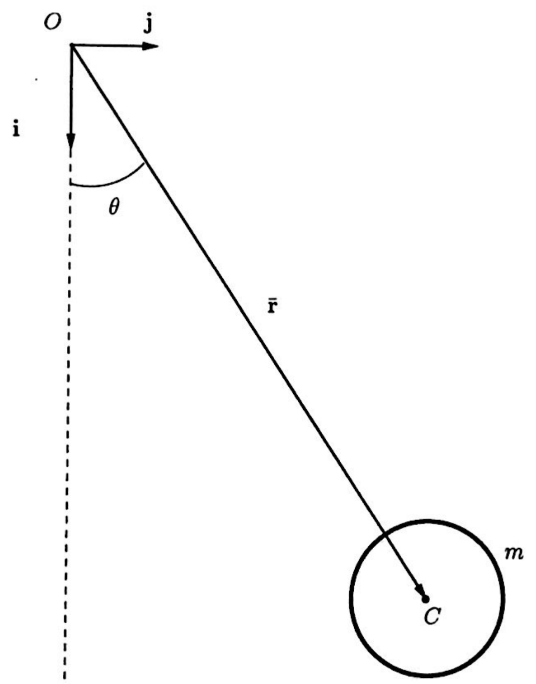

Figure 2

Place a corotational basis

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{pmatrix} \tag{22}$$

on the rod.The inertia tensor of the ball about its mass center is

$$\mathbf{I}^C = \frac{2}{5} m r^2 \, \mathbf{I} \ \mathrm{kg} \cdot \mathrm{m}^2. \tag{23}$$

(a) Apply the transfer theorem, namely

$$\mathbf{I}^O = \mathbf{I}^C + m \left[(\bar{\mathbf{r}} \cdot \bar{\mathbf{r}}) \, \mathbf{I} - \bar{\mathbf{r}} \otimes \bar{\mathbf{r}} \right], \tag{24}$$

to calculate $\mathbf{I}^O$.

(b) Apply Euler's law for angular momentum about O to obtain the differential equation for $\theta(t)$.

(c) Let $\mathbf{N} = N_1 \, \mathbf{e}_1 + N_2 \, \mathbf{e}_2 + N_3 \, \mathbf{e}_3$ be the force supplied by the axle. Calculate N_1, N_2, and N_3 as functions of θ and its time derivatives.

(d) Recall that the kinetic energy of the pendulum is given by

$$T = \frac{1}{2} I^O_{zz} \dot{\theta}^2 \tag{25}$$

Argue that energy is conserved for the pendulum, and write out its energy equation, taking the potential energy is taken to be zero in the stable equilibrium configuration.

Problem 8

Consider a rigid wheel that is a narrow disk. Suppose that it is rolling without slipping along a rough horizontal straight road. Let the origin O be at the point of contact of the wheel with the ground in the reference configuration. Let $\mathbf{i}$ point horizontally and to the right, and let $\mathbf{j}$ point vertically downwards. At time t, the position vector of the mass center C of the wheel is

$$\bar{\mathbf{r}} = x \, \mathbf{i} - r \mathbf{j}, \tag{26}$$

where r is the radius of the wheel and x is the horizontal displacement of C. The angular velocity of the wheel is $\boldsymbol{\omega} = \omega \, \mathbf{k} = \dot{\theta} \, \mathbf{k}$, and its angular acceleration is $\boldsymbol{\alpha} = \alpha \, \mathbf{k} = \dot{\omega} \, \mathbf{k}$. The moment of inertia of the wheel about the principal axis CZ is

$$I^C_{zz} = \frac{1}{2} m r^2 \tag{27}$$

where m is the mass of the wheel.

(a) Obtain expressions for the velocity $\bar{\mathbf{v}}$ and acceleration $\bar{\mathbf{a}}$ of C at time t.

(b) What is the kinematical relationship between $\dot{x}$ and $\dot{\theta}$?

(c) Recall that if A is any point on the wheel, then the velocity of A is given by

$$\mathbf{v}_A = \bar{\mathbf{v}} + \boldsymbol{\omega} \times (\mathbf{r}_A - \bar{\mathbf{r}}) . \tag{28}$$

Suppose that B is the particle on the wheel with position vector

$$\mathbf{r}_B = \bar{\mathbf{r}} + r\,\mathbf{i} \tag{29}$$

at time t. Calculate the velocity $\mathbf{v}_B$ of B as a function of $\dot{x}$.

(d) Recall that the kinetic energy of the wheel may be expressed in the form

$$T = \bar{T} + \tilde{T}, \tag{30}$$

where

$$\bar{T} = \frac{1}{2} m \bar{\mathbf{v}} \cdot \bar{\mathbf{v}}, \quad \tilde{T} = \frac{1}{2} I_{zz}^{C} \omega^2. \tag{31}$$

Calculate T as a function of $\dot{x}$.

(e) Consider an unpowered cart that has four disk wheels, each of mass m kg, and also has an additional mass of $70m$ kg. Suppose that the cart is travelling along a road $KLMN$ and that the wheels are all rolling without slipping. Let KL and MN be horizontal and let LM a gentle downward slope. Let h be the height of KL above MN in meters. Argue that the total mechanical energy E of the cart is conserved. If the cart is travelling at a constant speed v_0 along KL, what would its speed v be along ~~LN?~~ MN