Physics 7B Midterm 2 Solutions - Fall 2017
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Problem 1

(a) For Gauss' Law, one can show that the electric field of an infinitely long wire with linear charge density λ has magnitude

$$E = \frac{\lambda}{2\pi\epsilon_0 r}$$

The total electric field is the sum of the electric fields of the two wires, so

$$\vec{E} = \frac{\lambda}{2\pi\epsilon_0}\left(\frac{\hat{x}}{x} + \frac{\hat{y}}{y}\right)$$

(b) Since we are only moving in the y-direction, $d\vec{s} = \hat{y}dy$ and so

$$\Delta V = -\int_A^B \vec{E}\cdot d\vec{s} = -\int_a^{3a} E_y dy$$
$$= -\int_a^{3a} \frac{\lambda}{2\pi\epsilon_0 y} dy = \boxed{\frac{-\lambda}{2\pi\epsilon_0}\ln(3)}$$

(c) The work done can be found by applying the result of part(b) as follows:

$$W = \int \vec{F}\cdot d\vec{x} = \int_{3a}^{a} qE_x dx + \int_{3a}^{a} qE_y dy$$
$$= \boxed{\frac{-q\lambda}{\pi\epsilon_0}\ln(3)}$$

(d) If we take the negative integral of the electric field for a single wire, we find

$$V = -\int \frac{\lambda}{2\pi\epsilon_0 r} dr = \frac{-\lambda}{2\pi\epsilon_0}\ln(r) + C$$

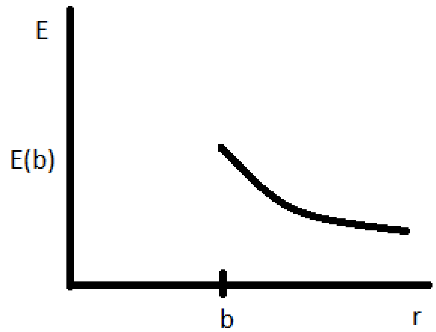
where C is a constant that defines the zero of the potential. Note that the logarithm blows up as $r \to \infty$, so we would need C to be infinite for the zero of potential to be at infinity. This would make the potential not helpful, so we do not do it.

Problem 2

(a) The electric field outside the sphere is just that of a point charge with charge Q:

$$E = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2} \text{ for } r > b$$

Inside the conductor, the electric field is zero. We have the following plot:



(b) Outside the sphere, we easily find

$$V = \frac{1}{4\pi\epsilon_0}\frac{Q}{r} \text{ for } r > b$$

Inside the sphere, the potential is constant and equal to the value at the surface of the conductor:

$$V = \frac{1}{4\pi\epsilon_0}\frac{Q}{b} \text{ for } r < b$$

(c) Here we use conservation of energy: $U_i = U_f + K$ so that

$$\frac{1}{4\pi\epsilon_0}\frac{-qQ}{2b} = \frac{1}{4\pi\epsilon_0}\frac{-qQ}{b} + \frac{1}{2}mv^2 \rightarrow \boxed{v = \sqrt{\frac{qQ}{4\pi\epsilon_0 bm}}}$$

(d) Inside the shell, there is no electric field, so the acceleration a is zero - $\boxed{a = 0}$

Problem 3

(a) We will consider the semicircle to be made of a charge density that varies with angle θ, which starts counterclockwise from the x-axis. The electric field in the x-direction of an infinitesimal amount of charge on the wire is $dE_x = -E\cos\theta$ so that

$$\begin{aligned} E_x &= -\int_0^{\pi} \frac{1}{4\pi\epsilon_0}\frac{\lambda(\theta)}{R^2}\cos\theta R d\theta \\ &= -\frac{1}{4\pi\epsilon_0 R}\left\{\int_0^{\pi/2} -\lambda\cos\theta d\theta + \int_{\pi/2}^{\pi} \lambda\cos\theta d\theta\right\} \\ &= \boxed{\frac{\lambda}{2\pi\epsilon_0 R}} \end{aligned}$$

(b) Following the same steps as above, we have $dE_y = -dE\sin\theta$ so

$$\begin{aligned}
E_y &= -\int_0^{\pi} \frac{1}{4\pi\epsilon_0}\frac{\lambda(\theta)}{R^2}\sin\theta R d\theta \\
&= -\frac{1}{4\pi\epsilon_0 R}\left\{\int_0^{\pi/2} -\lambda\sin\theta d\theta + \int_{\pi/2}^{\pi}\lambda\sin\theta d\theta\right\} \\
&= \boxed{0}
\end{aligned}$$

(c) We can answer this question using the calculations above. For the symmetry of the problem, we know that if the semicircle is composed entirely of positive charge, the electric field in the x direction should be zero: $\boxed{E_x = 0}$. The electric field in the y-direction should have the same magnitude as the field in part (a) but in the negative y-direction:

$$\boxed{E_y = -\frac{\lambda}{2\pi\epsilon_0 R}}$$

Problem 4

(a) The energy in a capacitor is given by $U = \frac{1}{2}CV^2$. Since the battery remains connected to the capacitor, we have initial and final energies

$$\begin{aligned}
U_i &= \frac{1}{2}V^2\left(\frac{\epsilon_0 A}{x}\right) \\
U_f &= \frac{1}{2}V^2\left(\frac{\epsilon_0 A}{2x}\right)
\end{aligned}$$

(b) The force between the plates is given by

$$F = -\frac{dU}{dx} = \frac{\epsilon_0 A V^2}{2x^2}$$

so that the work required is

$$W = \int_x^{2x} \frac{\epsilon_0 A V^2}{2x^2} dx = \frac{\epsilon_0 A V^2}{4x}$$

(c) To calculate the change in energy of the battery, we used the fact that the work done is equal to the change in energy of the battery plus the change in energy of the capacitor:

$$\begin{aligned}
W &= \Delta U_{cap} + \Delta U_{batt} \\
\Delta U_{batt} &= \frac{\epsilon_0 A V^2}{2x}
\end{aligned}$$

(d) If the battery is disconnected, then the charge on the capacitor remains constant while the potential changes. Then we can use $U = \frac{1}{2}\frac{Q^2}{C}$:

$$U_i = \frac{Q^2}{2}\frac{x}{\epsilon_0 A} = \frac{\epsilon_0 A V^2}{2x}$$
$$U_f = \frac{Q^2}{2}\frac{2x}{\epsilon_0 A} = \frac{\epsilon_0 A V^2}{x}$$

Where we have used $Q = \frac{\epsilon_0 A V}{x}$

Problem 5

(a) Each infinitesimally thin wafer contributes a resistance

$$dR = \frac{\rho dx}{\pi r^2(x)}$$

where $r(x)$ gives the radius as a function from the distance from the left side of the resistor:

$$r(x) = \left(\frac{b-a}{L}\right)x + a$$

then

$$R = \int_0^L \frac{\rho dx}{\pi r^2(x)} = \frac{\rho L}{\pi(b-a)}\left(\frac{1}{a} - \frac{1}{b}\right) = \boxed{\frac{\rho L}{\pi ab}}$$

(b) The current is the same through all of the surfaces. Otherwise, charge would build up inside the conductor.

(c) From the relation $E = \rho j$, we see that $\Phi_E = EA = \rho I$. Since I is constant through each surface, electric flux must be as well.

(d) The magnitude of the electric field decreases as we go from 1 to 2 since the current density decreases.