Physics 7B Midterm 2 Solutions - Fall 2017
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Problem 1

(a) Since the wire is a conductor, the electric field on the inside is simply zero. To find the electric field in the exterior of the wire, we use a cylindrical Gaussian surface of radius $r > a$ and length l. At a fixed distance $r > a$ from the wire, the electric field has constant magnitude as one goes up and down the length of the wire due to the symmetry of the wire. For the same reason, the electric field must point in the radial direction. Then the surface integral in Gauss' law is

$$\oiint \vec{E} \cdot d\vec{S} = |\vec{E}|(2\pi r l)$$

the total charge inside the Gaussian surface is $Q_{enc} = \lambda l$. Thus Gauss' law gives

$$|\vec{E}|(2\pi r l) = \frac{\lambda l}{\epsilon_0} \rightarrow \boxed{|\vec{E}| = \frac{\lambda}{2\pi\epsilon_0 r}}$$

(b) The potential difference as we go from ∞ to R, assuming $R > a$, is given by

$$\Delta V = -\int_{\infty}^{R} \frac{\lambda}{2\pi\epsilon_0 r} dr = -\frac{\lambda}{2\pi\epsilon_0} \left(\ln(R) - \ln(\infty)\right)$$

However, the natural logarithm diverges as you go to infinity, so the potential difference is infinite and not quite physical.

(c) Let us consider the case where the negatively charged wire goes through $x = 0$ and the positively charged wire goes through $x = d$. Then the electric field between the wires is just the sum of the electric fields of the wires:

$$\boxed{\vec{E} = -\frac{\lambda}{2\pi\epsilon_0 r}\hat{r} - \frac{\lambda}{2\pi\epsilon_0 (d-r)}\hat{r} = -\frac{\lambda d}{2\pi\epsilon_0 r(d-r)}\hat{r}}$$

Note that the above holds only for the interval $a < r < d - a$.

(d) To find the potential difference, we use the first form of the total electric field we found above:

$$\begin{aligned}\Delta V &= -\int_{a}^{d-a} \vec{E} \cdot \hat{r} dr = -\int_{a}^{d-a} \left\{-\frac{\lambda}{2\pi\epsilon_0 r} - \frac{\lambda}{2\pi\epsilon_0 (d-r)}\right\} dr \\ &= \frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{d-a}{a}\right) - \frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{a}{d-a}\right) \\ &= \boxed{\frac{\lambda}{\pi\epsilon_0} \ln\left(\frac{d-a}{a}\right)}\end{aligned}$$

(e) To find the capacitance per unit length, imagine that we have two wires of length l. The capacitance would then be

$$C = \frac{\lambda l}{\Delta V} = \frac{l\pi\epsilon_0}{\ln\left(\frac{d-a}{a}\right)} \to \boxed{\frac{C}{l} = \frac{\pi\epsilon_0}{\ln\left(\frac{d-a}{a}\right)}}$$

Problem 2

(a) We consider a spherical Gaussian surface with radius $r < a$. The total charge in the surface is found by performing a volume integral of the charge density:

$$Q_{enc} = \iiint \rho dV = 4\pi \int_0^r Ar^n r^2 dr = \frac{4\pi A}{n+3} r^{n+3}$$

Note that we have assumed that $n \neq -2$. Then from Gauss' Law, the electric field inside the sphere is given by

$$|\vec{E}|(4\pi r^2) = \frac{4\pi A}{\epsilon_0(n+3)} r^{n+3} \to |\vec{E}| = \frac{A}{\epsilon_0(n+3)} r^{n+1}$$

so we require $\boxed{n = -1}$

(b) Let us define the zero of electric potential to be at $r = \infty$. Then to find the electric potential outside the sphere, we need the total charge of the sphere:

$$Q_T = \frac{4\pi A a^2}{2}$$

The electric field outside the sphere is just that of a point charge with charge Q_T. So the electric potential for $r > a$ is

$$V(r) = \frac{1}{4\pi\epsilon_0}\frac{Q_T}{r} = \frac{Aa^2}{2\epsilon_0 r}$$

For $r < a$, the electric field is independent of r and taking the negative of the antiderivative of the electric field above gives

$$V(r) = -\frac{A}{2\epsilon_0} r + C$$

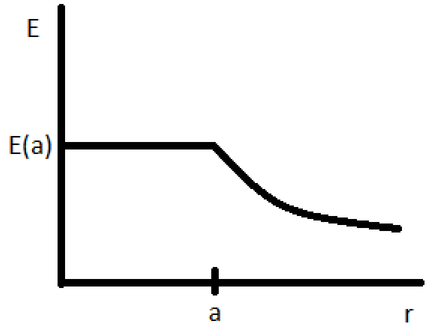
Where C is a constant of integration. To determine its value, we must match the potentials at $r = a$:

$$-\frac{Aa}{2\epsilon_0} + C = \frac{Aa}{2\epsilon_0} \to C = \frac{Aa}{\epsilon_0}$$

Thus the electric potential is given by

$$V(r) = \frac{Aa^2}{2\epsilon_0 r} \text{ for } r > a$$
$$V(r) = \frac{Aa}{\epsilon_0} - \frac{Ar}{2\epsilon_0} \text{ for } r < a$$

(c) For the electric field, we have



where $E(a) = \frac{A}{2\epsilon_0}$

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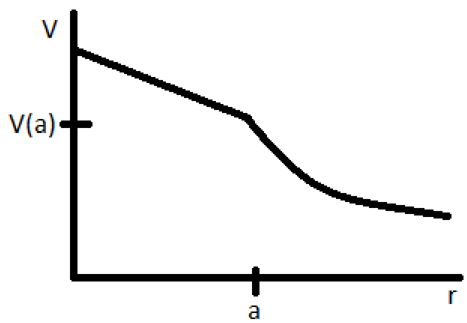


where $V(a) = \frac{Aa}{2\epsilon_0}$

Problem 3

(a) From the description, we have

$$\frac{\rho l_1}{\pi r_1^2} = \frac{\rho l_2}{\pi r_2^2} = \frac{2\rho l_1}{\pi r_2^2}$$

so

$$\frac{r_2}{r_1} = \sqrt{2}$$

and the ratio of the diameters is just $\sqrt{2}$.

(b) We think of breaking the spherical shell resistor into infinitely thin shells. Each of these shells has area $4\pi r^2$ and length dr, leading into an infinitesimal contribution the the overall resistance of the form

$$dR = \frac{\rho dr}{4\pi r^2}$$

so that the total resistance is

$$R = \int_{r_1}^{r_2} \frac{\rho dr}{4\pi r^2} = \frac{\rho}{4\pi}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) = \frac{1}{4\pi\sigma}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

Problem 4

(a) From the azimuthal symmetry of the annulus, we know that only the x-component of the electric field is nonzero: From the above diagram, we see that $dE_x = dE\cos\theta$ and that $\cos\theta = \frac{x}{\sqrt{x^2+r^2}}$ for a point on the annulus a distance r away from the center. Thus the contribution of an infinitesimal charge on the annulus is given by

$$\begin{aligned} dE_x &= \frac{1}{4\pi\epsilon_0}\frac{dq}{x^2+r^2}\left(\frac{x}{\sqrt{x^2+r^2}}\right) \\ &= \frac{1}{4\pi\epsilon_0}\frac{x\sigma r dr d\theta}{(x^2+r^2)^{3/2}} = \frac{1}{4\pi\epsilon_0}\frac{x\beta dr d\theta}{(x^2+r^2)^{3/2}} \end{aligned}$$

and so the electric field is found to be

$$\begin{aligned} E_x &= \int_0^{2\pi}\int_{R_1}^{R_2} \frac{1}{4\pi\epsilon_0}\frac{x\beta dr d\theta}{(x^2+r^2)^{3/2}} \\ &= \frac{\beta}{2\epsilon_0 x}\left(\frac{R_2}{\sqrt{x^2+R_2^2}} - \frac{R_1}{\sqrt{x^2+R_1^2}}\right) \end{aligned}$$

(b) We expand the above solution about $\frac{R_1}{x} \approx \frac{R_2}{x} \approx 0$ using the binomial expansion forumla:

$$(1+x)^n = 1 + nx + \cdots$$

so

$$\begin{aligned} E_x &\approx \frac{\beta}{2\epsilon_0 x}\left(\frac{R_2}{x}(1+\cdots) - \frac{R_1}{x}(1+\cdots)\right) \\ &= \frac{\beta(R_2-R_1)}{2\epsilon_0 x^2} \end{aligned}$$

Alternatively, one can state that at a large distance away from the annulus, the electric field must look like that of a point charge with charge equal to the total charge of the annulus:

$$\begin{aligned} Q_T &= \int_0^{2\pi}\int_{R_1}^{R_2} \sigma r dr d\theta \\ &= 2\pi\beta(R_2 - R_1) \end{aligned}$$

so that

$$E_x \approx \frac{1}{4\pi\epsilon_0}\frac{2\pi\beta(R_2-R_1)}{x^2}$$

Problem 5

(a) From the geometry of the set up, it is clear that the component of the electric field pointing in the vertical direction is zero. The total eletric field is then the sum of the electric fields in the x-direction:

$$\vec{E} = -\frac{2Q}{4\pi\epsilon_0\left(\frac{l^2}{4}+r^2\right)}\left(\frac{l}{2\sqrt{\frac{l^2}{4}+r^2}}\right)\hat{x}$$

$$= \boxed{-\frac{Ql}{4\pi\epsilon_0\left(r^2+\frac{l^2}{4}\right)^{3/2}}\hat{x}}$$

(b) We can find the electric potential by adding the potential of both charges:

$$V = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r}-\frac{1}{r+\Delta r}\right) = \frac{Q}{4\pi\epsilon_0}\frac{\Delta r}{r+\Delta r}$$

For $r \gg l$, $\Delta r \approx l\cos\theta$, so

$$V = \frac{Q}{4\pi\epsilon_0}\frac{l\cos\theta}{r^2}$$

Problem 6

(a) The work is takes to charge a capacitor plate to charge Q is given by

$$W = \int_0^Q V dq = \frac{1}{C}\int_0^Q q dq = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}CV^2$$

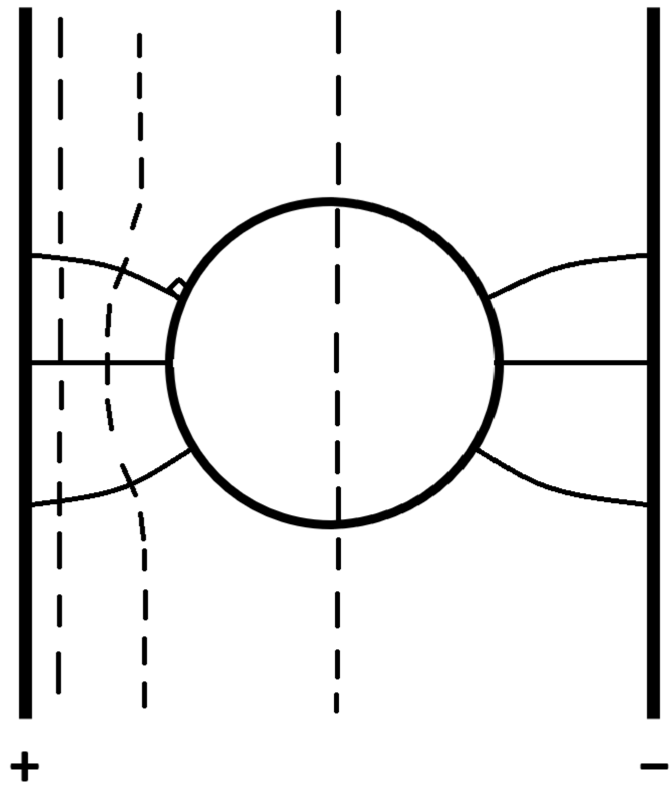
we intepret this work as being stored inside the capacitor, so $W = U$, where U is the internal energy of the capacitor.

(b) To get the energy density, we first divide both sides by the volume between the plates $V = Ad$.

$$u = \frac{U}{V} = \frac{1}{2Ad}\frac{Q^2}{\frac{\epsilon_0 A}{d}} = \frac{1}{2}\epsilon_0 E^2$$

where we have used $C = \frac{\epsilon_0 A}{d}$ and $E = \frac{Q}{\epsilon_0 A}$.

(c) We have



Where the solid lines are electric field lines and the dashed lines denote equipotential surfaces.