1. (20 points) Let $A = \begin{bmatrix} 1 & -1 & 2 & 2 \\ -2 & 3 & -3 & -1 \\ 3 & -3 & 6 & 7 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix}$.

a) Find all solutions to $A\vec{x} = \vec{0}$ in the parametric vector form.

b) Do the same for $A\vec{x} = \vec{b}$.

c) Do the columns of A span $\mathbb{R}^3$? Justify your answer.

d) Are the columns of A linearly independent? Justify your answer.

A:

Write down the augmented matrix

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & 2 & 0 \\ -2 & 3 & -3 & -1 & 0 \\ 3 & -3 & 6 & 7 & 0 \end{array}\right].$$

Perform row elimination and obtain REF (one possible form)

$$\left[\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right].$$

We find that x_3 is a free variable.

The solution set of the corresponding homogeneous equation is

$$x_3 \begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \end{bmatrix}.$$

b)

Same as a), write down the augmented matrix

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & 2 & 1 \\ -2 & 3 & -3 & -1 & -3 \\ 3 & -3 & 6 & 7 & 3 \end{array}\right].$$

The REF is

$$\left[\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right].$$

A special solution is $\begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}$.

The parametric solution is

$$\begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \end{bmatrix}.$$

c) Yes. In a) we find that the REF of A has a pivot in every row, so the columns of A span $\mathbb{R}^3$.

d) No. Since the equation $A\vec{x} = 0$ has a non-trivial solution, the columns of A are not linearly independent.

2. (15 points) True or False: If True, explain why. If False, give an explicit numerical example for which the statement does not hold.

a) Let A and B be n by n matrices such that A is invertible and B is not invertible. Then, AB is not invertible.

b) Let $\vec{v}_1, \vec{v}_2, \vec{v}_3$ be vectors in $\mathbb{R}^n$. If $\{\vec{v}_1, \vec{v}_2\}, \{\vec{v}_1, \vec{v}_3\}$, and $\{\vec{v}_2, \vec{v}_3\}$ are each linearly independent sets, then $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a linearly independent set.

c) If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, and $ad - bc \neq 0$, then A is invertible.

A:

a) This is true.

Since A is invertible, A^{-1} is invertible as well. Assume AB is invertible, then $A^{-1}(AB) = B$ is also invertible. But this conflicts with the assumption that B is not invertible.

b) This is false.

A simple example is in $\mathbb{R}^2$, where $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. When you adjoin any two of these vectors in a 2 by 2 matrix, you are either in an echelon form with 2 pivots, or just a single row swap away. But, any linearly independent set in $\mathbb{R}^2$ can have at most two vectors in it because the dimension of $\mathbb{R}^2$ is 2, so $\{v_1, v_2, v_3\}$ is linearly dependent.

c) This is true.

First a, b cannot be both zero, otherwise $ad - bc = 0$. Assume $a \neq 0$ (otherwise we can exchange the first and the second row, and the condition is the same as $bc - ad \neq 0$), the REF is $A = \begin{bmatrix} a & c \\ 0 & d - bc/a \end{bmatrix}$. The REF has two pivots if and only if $ad - bc \neq 0$. Hence A is invertible if and only if $ad - bc \neq 0$.

3. (5 points) Compute the matrix inverse of $A = \begin{bmatrix} 1 & -2 & 3 \\ -2 & 3 & -4 \\ 2 & 0 & -1 \end{bmatrix}$

A:
Consider the augmented matrix $[A|I]$
Perform row reduction and convert the left part to the RREF.
The final answer is $A^{-1} = \begin{bmatrix} 3 & 2 & 1 \\ 10 & 7 & 2 \\ 6 & 4 & 1 \end{bmatrix}$.

4. (10 points) a) In $\mathbb{R}^2$, the operation of rotating a vector by an angle θ along the counter clockwise direction is a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$. Write down A, the standard matrix for T.

b) Let B be the standard matrix for rotation by an angle ϕ and let C be the standard matrix for rotation by the angle $(\theta + \phi)$, both along the counter clockwise direction. Write down the matrix B and C. Verify that $AB = C$

A:
a) $T(e_1) = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$, $T(e_2) = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$.
Hence the standard matrix is
$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

b) Similar to a), we find $B = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$ and $C = \begin{bmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{bmatrix}$.

$$AB = \begin{bmatrix} \cos\theta\cos\phi - \sin\theta\sin\phi & -\cos\theta\sin\phi - \sin\theta\cos\phi \\ \sin\theta\cos\phi + \cos\theta\sin\phi & -\sin\theta\sin\phi + \cos\theta\cos\phi \end{bmatrix}.$$

Use the relation $\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$ and $\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$ we find $C = AB$.