1. (10 points) Compute the determinant $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix}$

A: Perform elementary row reduction. One possible route is (5 points)

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 0 & -2 \\ 0 & -2 & 0 \end{vmatrix} = 4 \begin{vmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{vmatrix}.$$

Perform cofactor expansion for the (1,1) element (3 points)

$$= (-1)^{1+1} 4 \begin{vmatrix} 0 & -1 \\ -1 & 0 \end{vmatrix}$$

The final answer is -4. (2 points)

2. True or False (15 points) If True, explain why. If False, give a counterexample.

(a) Let A be an $n \times n$ matrix. If two columns of A are the same, then the determinant $\det A = 0$.

A: True. (2 points)If two columns of A are the same, then the columns of A are linearly dependent. So A is not invertible and $\det A = 0$. (3 points)

(b) If vectors $\vec{u}, \vec{v} \in \mathbb{R}^2$ and $\vec{u} \cdot \vec{v} = 1$, then $\{\vec{u}, \vec{v}\}$ is also a basis for $\mathbb{R}^2$.

A: False. (2 points) Consider $\vec{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{v} = \vec{u}$. Then $\vec{u} \cdot \vec{v} = 1$ is satisfied, but $\vec{u}, \vec{v}$ are linearly dependent and cannot be a basis. (3 points)

(c) If the $n \times n$ matrix A is the matrix representation of a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ with respect to one basis, and B is the matrix representation of the same linear transformation with respect to a different basis, then $\det(A) = \det(B)$.

A: True. (2 points) We have $A = P^{-1}BP$, where P is the invertible matrix representing the change of basis. Then $\det(A) = \det(P^{-1}) \det(B) \det(P) = \det B$. (3 points)

3. (15 points) Let $T : \mathbb{P}_2 \to \mathbb{P}_2$ by $p(x) \mapsto (x \cdot p(x))'$. (Note $\frac{d}{dx} f(x) \equiv f'(x) \equiv (f(x))'$).

(a) Write out the matrix representation $[T]_B$ of this transformation with respect to the basis $B = \{1, 2x, x^2 - 1\}$.

A: Since that $T(1) = (x)' = 1, T(2x) = (2x^2)' = 4x$ and $T(x^2 - 1) = (x^3 - x)' = 3x^2 - 1$. (3 points) Expressing these as coordinate vectors, we get

$$[T(1)]_B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, [T(x)]_B = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, [T(x^2)]_B = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}.$$

(3 points) Putting them all together in the same matrix gives

$$[T]_B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

(4 points)

(b) Evaluate the B-coordinate of x^2+3, and **use your matrix from (a)** to find $T(x^2+3)$.

A: $[x^2+3]_B = \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$, (2 points) so

$$[T(x^2+3)]_B = [T]_B[x^2+3]_B = \begin{bmatrix} 6 \\ 0 \\ 3 \end{bmatrix}.$$

Thus, $T(x^2+3) = 6 + 3(x^2-1) = 3x^2+3$. (3 points)

4. (15 points) Let $V = \mathbb{R}^2$, $B = \left\{ \begin{bmatrix} -1 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ -7 \end{bmatrix} \right\}$, and $C = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.

(a) Find the change of basis matrix $\underset{C \leftarrow B}{P}$.

A: Find $\underset{C \leftarrow B}{P}$ by using (3 points)

$$\underset{C \leftarrow B}{P} = \underset{C \leftarrow \mathcal{E}}{P}\underset{\mathcal{E} \leftarrow B}{P} = \left(\underset{\mathcal{E} \leftarrow C}{P}\right)^{-1}\underset{\mathcal{E} \leftarrow B}{P}.$$

Since (2 points)

$$\underset{\mathcal{E} \leftarrow C}{P} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}, \quad \underset{\mathcal{E} \leftarrow B}{P} = \begin{bmatrix} -1 & 1 \\ 8 & -7 \end{bmatrix},$$

Then (3 points)

$$\left(\underset{\mathcal{E} \leftarrow C}{P}\right)^{-1} = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}.$$

we have (2 points)

$$\underset{C \leftarrow B}{P} = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 8 & -7 \end{bmatrix} = \begin{bmatrix} 9 & -8 \\ -10 & 9 \end{bmatrix}.$$

(b) Use the change of basis matrix $\underset{C \leftarrow B}{P}$ obtained in a) to express $\vec{x} = \begin{bmatrix} 1 \\ -7 \end{bmatrix}$ as a linear combination of the vectors in C.

A: First compute $[\vec{x}]_B$. In this case, we directly obtain (3 points)

$$[\vec{x}]_B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Then (2 points)

$$[\vec{x}]_C = \underset{C \leftarrow B}{P} [\vec{x}]_B = \begin{bmatrix} -8 \\ 9 \end{bmatrix}.$$

5. (15 points) Diagonalize the following matrices, if possible:

(a) $\begin{bmatrix} 3 & 2 \\ 0 & 3 \end{bmatrix}$

A: $\det(A - \lambda I) = (3 - \lambda)^2 - 0$, so we get $\lambda = 3$ as a double root. (2 points) We see that $A - 3I = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$ only has a one-dimensional null space. Therefore, as the dimension of an eigenspace (geometric multiplicity) does not match the multiplicity of the corresponding root (algebraic multiplicity). (2 points) Hence A is not diagonalizable.(2 points)

(b) $\begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}.$

A: (3 points)

$$\begin{aligned}
\det(A - \lambda I) &= (3 - \lambda)[(3 - \lambda)^2 - 1] - [(3 - \lambda) - 1] + [1 - (3 - \lambda)] \\
&= (3 - \lambda)[\lambda^2 - 6\lambda + 9 - 1] - 3 + \lambda + 1 + 1 - 3 + \lambda \\
&= -\lambda^3 + 9\lambda^2 - 24\lambda + 20 \\
&= -(\lambda - 5)(\lambda - 2)^2
\end{aligned}$$

Now,

$$A - 2I = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

which has a null space spanned by $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$. (2 points)

$$A - 5I = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix},$$

which has a null space spanned by $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. (2 points)

Thus, $A = PDP^{-1}$, where

$$P = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}, D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}.$$

(2 points)