ME 132, Fall 2015, Quiz # 2

# 1	# 2	# 3	# 4	# 5	# 6	Total	NAME
14	10	8	6	14	8	60	

Rules:

1. 2 sheets of notes allowed, 8.5×11 inches. Both sides can be used.
2. Calculator is allowed. **Keep it in plain view on the desk next to you.**
3. **No laptops, phones, headphones, pads, tablets, or any other such device may be out. If such a device is seen after 1:10PM, your test will be confiscated, and you will get a 0 for the exam.**
4. **Sit with at least one open space between every student**
5. **The exam ends promptly at 2:00 PM**.
6. **Stop working, and turn in exams when notified.**

Facts:

1. **3rd order stability test:** The roots of the third-order polynomial

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3$$

all have negative real-parts if and only if $a_1 > 0, a_3 > 0$ and $a_1 a_2 > a_3$.

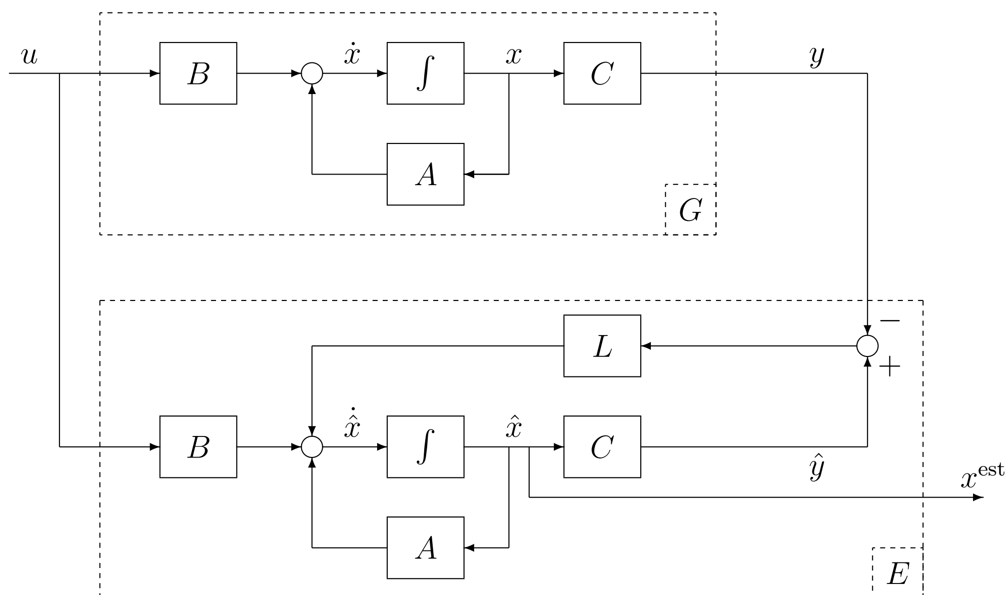
2. The characteristic polynomial of the 1st order (vector) differential equation $\dot{x}(t) = Ax(t)$ is $\det(\lambda I_n - A)$, where n is the dimension of x.

1. **Remark: This is not a hard problem. Do not be scared off by the long description**. A *state-estimator*, E, is a dynamical system that attempts to estimate the internal state x of a system G, by observing only the inputs, u, and outputs, y, of G. The estimator <u>does not</u> have access to the initial condition, $x(0)$ of the process. The system G is assumed to be governed by a known, linear, state-space model, namely

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned}$$

Since A, B, C are known, and the estimator has access to u and y, the strategy is that the estimator should be a mathematical *copy* of the process G, with state $\hat{x}$. The goal is to get $\hat{x}$ to converge to x, asymptotically in time (regardless of u). Hence, it makes sense that the estimator's input should be the same input as the process (u), and the state, $\hat{x}$, inside the estimator, should be adjusted in some manner proportional to the difference between the process output y, and the estimator's prediction of y, in a way that makes $\hat{x}(t) - x(t) \to 0$ as $t \to \infty$, regardless of u and $x(0)$.

A diagram of such a system is shown below. Note that the estimator E has two inputs, "receiving" both u and y, and produces one output, namely producing the estimate, $x^{\text{est}} = \hat{x}$ of x.



All boxed quantities, A, B, C, L, F are matrices. The boxes labeled $\int$ are integrators (note the signal definitions for x and $\hat{x}$ and their time derivatives). All summing junction sign conventions, unless otherwise marked, are positive.

(a) Define

$$z(t) := \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix}$$

Fill in the matrices below, to complete the state-space model which governs the entire (process and estimator) system

$$\dot{z}(t) = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} z(t) + \begin{bmatrix} \quad \\ \quad \end{bmatrix} u(t)$$

and

$$x^{\text{est}}(t) = \begin{bmatrix} \quad & \quad \end{bmatrix} z(t)$$

(b) Define the *estimation error*, e as $e(t) := x(t) - \hat{x}(t)$. Show that e satisfies a simple, linear differential equation, whose right-hand side only involves, A, C, L and e.

(c) What is one important property that the matrix L must have in order for $\hat{x}(t) - x(t) \rightarrow 0$ as $t \rightarrow \infty$, regardless of u, $x(0)$ and $\hat{x}(0)$?

(d) What are some issues that are not addressed here, that might impact (and distinguish among) several various choices for L?

2. Suppose $0 < \xi < 1$ and $\omega_n > 0$. Consider the system

$$\ddot{y}(t) + 2\xi\omega_n\dot{y}(t) + \omega_n^2 y(t) = \omega_n \dot{u}(t)$$

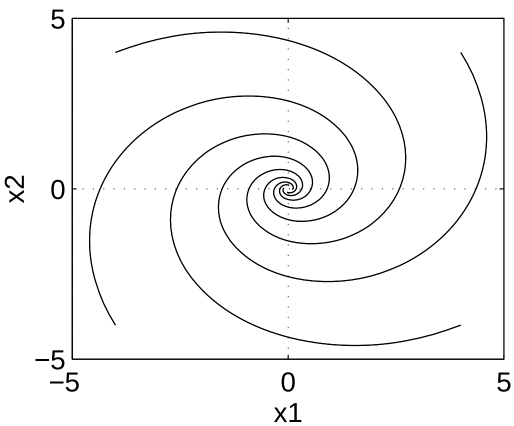
subject to the initial conditions $y(0^-) = 0, \dot{y}(0^-) = 0$, and the unit-step forcing function, namely $u(t) = 0$ for $t = 0$, and $u(t) = 1$ for $t > 0$. Show that the response is

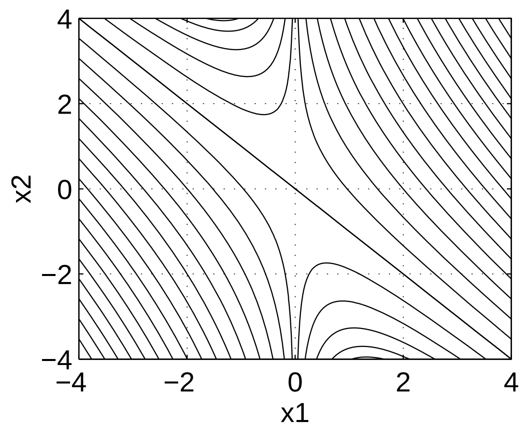
$$y(t) = \frac{1}{\sqrt{1-\xi^2}} e^{-\xi\omega_n t} \sin\left(\sqrt{1-\xi^2}\,\omega_n t\right)$$

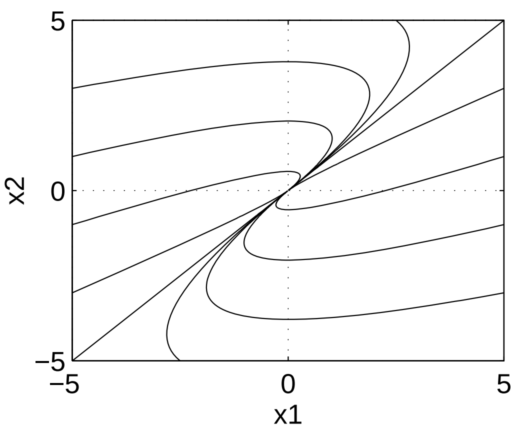
Hint: Recall that the set of all real-valued homogeneous solutions of $\ddot{y}(t) + 2\xi\omega_n\dot{y}(t) + \omega_n^2 y(t) = 0$ is $y_H(t) = Ae^{-\xi\omega_n t}\cos\left(\sqrt{1-\xi^2}\,\omega_n t\right) + Be^{-\xi\omega_n t}\sin\left(\sqrt{1-\xi^2}\,\omega_n t\right)$ where A and B are any real numbers.

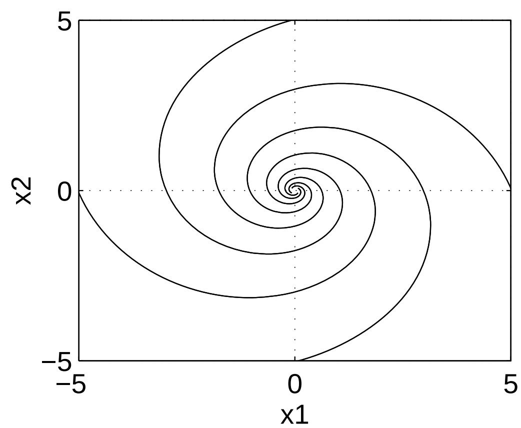
3. Consider the 2-state system governed by the equation $\dot{x}(t) = Ax(t)$. Shown below are the phase-plane plots ($x_1(t)$ vs. $x_2(t)$) for 4 different cases. Match the plots with the A matrices, and correctly draw in arrows indicating the evolution in time.

$$A_1 = \begin{bmatrix} -2 & 0 \\ 3 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} -3 & 2 \\ -1 & 0 \end{bmatrix}, \quad A_4 = \begin{bmatrix} -1 & 3 \\ -3 & -1 \end{bmatrix}$$









4. (a) What is the general form of the solution to the differential equation

$$\ddot{x}(t) + 6\dot{x}(t) + 5x(t) = 0$$

(b) What is the general form of the solution to the differential equation

$$\ddot{x}(t) + 6\dot{x}(t) + 5x(t) = -10$$

Your expressions both should have two free constants.

5. A first-order process, with state x, input u, disturbance d and output y is governed by

$$\dot{x}(t) = x(t) + u(t) + d(t), \qquad y(t) = x(t)$$

(a) Is the process stable?

(b) Suppose $x(0) = -3$, and $u(t) = d(t) \equiv 0$ for all $t \geq 0$. What is the solution $y(t)$ for $t \geq 0$.

(c) A PI (Proportional plus Integral) controller is proposed

$$\begin{aligned} u(t) &= K_P \left[r(t) - y(t) \right] + K_I z(t) \\ \dot{z}(t) &= r(t) - y(t) \end{aligned}$$

Define the state-vector q as

$$q(t) := \begin{bmatrix} x(t) \\ z(t) \end{bmatrix}$$

Write the state-space model of the closed-loop system, with state q, inputs (d, r) and outputs (y, u).

(d) What is the closed-loop characteristic polynomial?

(e) For what values of K_P and K_I is the closed-loop system stable?

(f) The closed-loop system is 2nd order. What are the appropriate values of K_P and K_I so that the closed-loop system eigenvalues are described by $\xi = 0.8, \omega_n = 0.5$?

(g) What are the appropriate values of K_P and K_I so that the closed-loop system eigenvalues are described by $\xi = 0.8, \omega_n = 1.0$?

(h) What are the appropriate values of K_P and K_I so that the closed-loop system eigenvalues are described by $\xi = 0.8, \omega_n = 2.0$?

6. A 2nd-order, unstable process, with control input u, disturbance input d, and output y, is governed by the equation

$$\ddot{y}(t) + \dot{y}(t) - y(t) = u(t) + d(t)$$

A PI (Proportional plus Integral) controller is proposed, both to stabilize the system, and provide good disturbance rejection,

$$\begin{aligned} u(t) &= K_P\left[r(t) - y(t)\right] + K_I z(t) \\ \dot{z}(t) &= r(t) - y(t) \end{aligned}$$

Here r is a reference input.

(a) Using just the controller equations, express $\dot{u}(t)$ in terms of $r, \dot{r}, y$ and $\dot{y}$.

(b) By differentiating the process equation, and substituting, derive the **closed-loop differential equation** relating r and d (and possibly their derivatives) to the output variable y (and its derivatives). The variable u should not appear in these equations.

(c) Using the 3rd-order test for stability, determine the conditions on K_P and K_I such that the closed-loop system is stable.