Math 54	**Name:**	______________________
Fall 2017		
Exam 2	**Student ID**:	______________________
10/31/17		
Time Limit: 80 Minutes	**GSI or Section**:	______________________

This exam contains 6 pages (including this cover page) and 7 problems. Problems are printed on both sides of the pages. Enter all requested information on the top of this page.

> This is a closed book exam. No notes or calculators are permitted.
> We will drop your lowest scoring question for you.

You are required to show your work on each problem on this exam. **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

If you need more space, there are blank pages at the end of the exam. Clearly indicate when you have used these extra pages for solving a problem. However, it will be greatly appreciated by the GSIs when problems are answered in the space provided, and your final answer **must** be written in that space. Please do not tear out any pages.

Do not write in the table to the right.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
Total:	70	

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1. (10 points) Let $V = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4 \middle| x_1 + x_2 + 2x_3 = 0 \right\}$ and $W = \text{Nul} \begin{bmatrix} 0 & 1 & 2 & 0 \\ 0 & 4 & 6 & 2 \end{bmatrix}$.

(a) (3 points) What is the dimension of V?

Solution: V is in fact the null space of

$$\begin{bmatrix} 1 & 1 & 2 & 0 \end{bmatrix}.$$

We see that there is only one pivot and three free variables. Thus, the dimension of V is 3.

(b) (3 points) What is the dimension of W?

Solution: The matrix row reduces to

$$\begin{bmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & -2 & 2 \end{bmatrix}.$$

which has two pivots and two free variables. Thus, the dimension of W is 2.

(c) (4 points) What is the dimension of the intersection of V and W?

Solution: The intersection of these two null spaces will be the null space of the matrix below with rows coming from both matrices.

$$\begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 4 & 6 & 2 \end{bmatrix}$$

We can row reduce this matrix to

$$\begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -2 & 2 \end{bmatrix}$$

which shows that there are three pivots and only one free variable. Thus, the dimension of the intersection of V and W (the null space of this matrix) is 1.

2. (a) (5 points) Find all values of c such that the following matrix is diagonalizable

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & -1 & c \\ 0 & 0 & -1 \end{bmatrix}$$

Solution: The matrix is upper triangular so its eigenvalues are the elements of the diagonal. The only repeated eigenvalue is -1 with multiplicity 2. The corresponding eigenspace is

$$E_{-1} = \mathrm{Nul}(A - (-1)I) = \mathrm{Nul} \begin{bmatrix} 2 & 2 & 2 \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix}$$

which has dimension 2 only when $c = 0$. Therefore the matrix A is diagonalizable only when $c = 0$.

(b) (5 points) For the values of c found in part (a), find an invertible matrix P and diagonal matrix D such that $A = PDP^{-1}$.

Solution: We need to find a basis of eigenvectors when $c = 0$. We have

$$E_1 = \mathrm{Nul}(A - I) = \mathrm{Nul} \begin{bmatrix} 0 & 2 & 2 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} = \mathrm{Nul} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \mathrm{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$E_{-1} = \mathrm{Nul}(A + I) = \mathrm{Nul} \begin{bmatrix} 2 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathrm{Span} \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

We can then write $A = PDP^{-1}$ where

$$P = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

3. (10 points) Label the following statements as True or False. The correct answer is worth 1 point and a brief justification is worth 1 point. Credit for the justification can only be earned in conjunction with a correct answer. No points will be awarded if it is not clear whether you intended to mark the statement as True or False.

(a) (2 points) If A and B are 2×2 matrices with $\det(A) = \det(B) = 0$, then $\{A, B\}$ is linearly dependent in the vector space of 2×2 matrices, $M_{2\times 2}$.

Solution: False.
For example, take $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then, $c_1A + c_2B = 0$ implies $c_1 = c_2 = 0$. In fact, A and B are part of the standard basis for $M_{2\times 2}$.

(b) (2 points) If A is an $n\times n$ matrix and 0 is an eigenvalue of A, then A is not diagonalizable.

Solution: False.
The matrix $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ has 0 as an eigenvalue but is diagonalizable (it is already diagonal).

(c) (2 points) Suppose $u, v, w \in \mathbb{R}^3$. If u is orthogonal to v and v is orthogonal to w, then u is orthogonal to w.

Solution: False.
Consider $u = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $v = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $w = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. Then, $u \cdot v = v \cdot w = 0$, but $u \cdot w = 1 \neq 0$.

(d) (2 points) If A and B are diagonalizable 2×2 matrices, then $A + B$ is diagonalizable.

Solution: False.
For example $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ are both diagonalizable since they have two distinct eigenvalues. However $A + B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is not since the only eigenvalue is 1 and its eigenspace E_1 is spanned by $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Thus, $\dim(E_1) = 1$ so there is no basis of $\mathbb{R}^2$ consisting of eigenvectors of $A + B$.

(e) (2 points) Suppose $\mathcal{B}$ and $\mathcal{C}$ are bases of $\mathbb{R}^n$. If $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$ and $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ are the change of coordinates matrices from $\mathcal{C}$ to $\mathcal{B}$ and from $\mathcal{B}$ to $\mathcal{C}$, respectively, then $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}\,\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ is the $n \times n$ identity matrix.

Solution: True.
$\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}\,\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = P_{\mathcal{B}}P_{\mathcal{C}}^{-1}P_{\mathcal{C}}P_{\mathcal{B}}^{-1} = P_{\mathcal{B}}P_{\mathcal{B}}^{-1} = I_n$ where $P_{\mathcal{B}}$ and $P_{\mathcal{C}}$ are the matrices of the coordinate transformations.

4. (a) (2 points) State the rank theorem for an $m \times n$ matrix A. (*Hint:* This theorem might be useful in both parts (b) and (c).)

Solution: The rank theorem states that

$$n = \text{Rank}(A) + \dim \text{Nul}(A).$$

(b) (4 points) Suppose that B is a 3×3 diagonalizable matrix whose characteristic polynomial is $\lambda^2(1-\lambda)$. Find the rank of B.

Solution: Since B is diagonalizable, the dimension of the eigenspace E_0 is 2 (the multiplicity of 0 as an eigenvalue). But, E_0 is simply the null space of A. Thus, the rank theorem gives

$$\text{Rank}(B) = 3 - 2 = 1.$$

(c) (4 points) Suppose that C is a 4×4 matrix such that $C^2 = 0$. Show that $\text{Rank}(C) \leq 2$. (*Hint:* Show that $\text{Col}(C) \subset \text{Nul}(C)$ and look at the hint in part (a).)

Solution: Suppose that $\{w_1, \ldots, w_k\} \subset \mathbb{R}^4$ is a basis of the column space of C so that $\text{Rank}(C) = k$. Then, each w_i is equal to Cv_i for some $v_i \in \mathbb{R}^4$. Since $C^2 = 0$,

$$Cw_i = C^2 v_i = 0$$

for all i. Thus, $\{w_1, \ldots, w_k\} \subset \text{Nul}(C)$ so $\dim \text{Nul}(C) \geq k$. The rank theorem then gives

$$4 = k + \dim \text{Nul}(C) \geq 2k$$

so $k \leq 2$ as desired.

5. (10 points) Consider the subspace of $\mathbb{R}^4$ below.

$$W = \text{Span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \right\}$$

(a) (5 points) Apply the Gram-Schmidt algorithm to produce an orthogonal basis of W.

Solution: W is given to us as $\text{Span}\{v_1, v_2, v_3\}$ so we apply the algorithm to $\{v_1, v_2, v_3\}$ to get an orthogonal basis $\{u_1, u_2, u_3\}$.

$$u_1 = v_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$u_2 = v_2 - \frac{v_2 \cdot u_1}{u_1 \cdot u_1} u_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$u_3 = v_3 - \frac{v_3 \cdot u_1}{u_1 \cdot u_1} u_1 - \frac{v_3 \cdot u_2}{u_2 \cdot u_2} u_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} - \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 3 \\ 4 \end{bmatrix}$$

Thus, the resulting orthogonal basis is

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \\ 4 \end{bmatrix} \right\}.$$

(b) (5 points) Find the distance between $v = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 2 \end{bmatrix}$ and W.

Solution: The distance is given by $\|v - \text{Proj}_W(v)\|$ where Proj_W is orthogonal projection onto W. We have

$$\text{Proj}_W(v) = \frac{v \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{v \cdot u_2}{u_2 \cdot u_2} u_2 + \frac{v \cdot u_3}{u_3 \cdot u_3} u_3 = \begin{bmatrix} 1 \\ 1 \\ 3/5 \\ 4/5 \end{bmatrix}$$

so $\|v - \text{Proj}_W(v)\| = \sqrt{\left(\frac{8}{5}\right)^2 + \left(\frac{6}{5}\right)^2} = 2.$

6. (a) (5 points) Show that

$$\mathcal{B} = \{1 + 5t^2, 2 + t + 6t^2, 3 + 4t\}$$

is a basis for the vector space $\mathbb{P}_2$ of polynomials of degree less than or equal to 2.

Solution: We wish to show that these polynomials are linearly independent and span $\mathbb{P}_2$. It is enough to check that the coordinates of these polynomials in the standard basis $\mathcal{S} = \{1, t, t^2\}$ are a basis of $\mathbb{R}^3$. The coordinates are

$$[1 + 5t^2]_{\mathcal{S}} = \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix} \qquad [2 + t + 6t^2]_{\mathcal{S}} = \begin{bmatrix} 2 \\ 1 \\ 6 \end{bmatrix} \qquad [3 + 4t]_{\mathcal{S}} = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}$$

so we only need to show that

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}$$

is invertible. The matrix is invertible since

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & -4 & -15 \end{vmatrix} = \begin{vmatrix} 1 & 4 \\ -4 & -15 \end{vmatrix} = -15 + 16 = 1 \neq 0.$$

(b) (5 points) Compute the change of coordinates matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ from $\mathcal{B}$ to $\mathcal{C}$ where $\mathcal{B} = \{1+5t^2, 2+t+6t^2, 3+4t\}$ is as in part (a) and $\mathcal{C}$ is the following basis of $\mathbb{P}_2$.

$$\mathcal{C} = \{1+t, t+t^2, 1\}$$

Solution: Recall that the columns of $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ are $[b_1]_\mathcal{C}, [b_2]_\mathcal{C}$, and $[b_3]_\mathcal{C}$ where b_1, b_2, and b_3 are the vectors in $\mathcal{B}$. We find all these coordinates simultaneously (using $\mathcal{S}$) as follows.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 2 & 3 \\ 1 & 1 & 0 & 0 & 1 & 4 \\ 0 & 1 & 0 & 5 & 6 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 2 & 3 \\ 0 & 1 & -1 & -1 & -1 & 1 \\ 0 & 1 & 0 & 5 & 6 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 2 & 3 \\ 0 & 1 & -1 & -1 & -1 & 1 \\ 0 & 0 & 1 & 6 & 7 & -1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -5 & -5 & 4 \\ 0 & 1 & 0 & 5 & 6 & 0 \\ 0 & 0 & 1 & 6 & 7 & -1 \end{array}\right]$$

Thus,

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} -5 & -5 & 4 \\ 5 & 6 & 0 \\ 6 & 7 & -1 \end{bmatrix}.$$

7. (a) (5 points) If A is a 3×3 matrix with real entries such that 2 and $3 - i$ are eigenvalues of A, compute $\det(A)$. (Note: It is **NOT** enough to get full credit if you only compute the determinant on a specific example that has these eigenvalues.)

Solution: Since A has real entries, its non-real eigenvalues come in complex conjugate pairs. Thus, the third eigenvalue of A must be $3 + i$. Further, the determinant of A is the product of all the eigenvalues of A. This gives

$$\det(A) = 2(3 + i)(3 - i) = 20.$$

(b) (5 points) Suppose that A and B are $n \times n$ matrices and B is invertible. Show that there is a $\lambda \in \mathbb{C}$ such that $A + \lambda B$ is not invertible.

Solution: Since $A + \lambda B$ is square, the invertible matrix theorem tells us that $A + \lambda B$ is invertible if and only if its null space is trivial. Thus, we want to show that there is a $\lambda \in \mathbb{C}$ and $v \neq 0$ such that

$$(A + \lambda B)v = 0.$$

Multiplying by B^{-1} on the left gives

$$(B^{-1}A + \lambda I)v = 0.$$

Thus, our problem is reduced to the fact that $B^{-1}A$ has an eigenvalue in $\mathbb{C}$. We know (via the fundamental theorem of algebra) that every matrix has at least one eigenvalue in $\mathbb{C}$ so we are done.

Extra space.

Extra space.