EECS C128/ ME C134
Final
Thu. May 14, 2015
1510-1800 pm

Name: Key.
SID:

- Closed book. One page, 2 sides of formula sheets. No calculators.
- There are 8 problems worth 100 points total.

MAX = 94
MIN = 16
mean = 52.7/100
σ = 17.4
median = 54

Problem	Points	Score
1	14	
2	14	
3	16	
4	8	
5	13	
6	13	
7	14	
8	8	
Total	100	

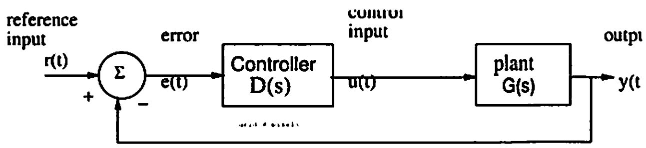
In the real world, unethical actions by engineers can cost money, careers, and lives. The penalty for unethical actions on this exam will be a grade of 'F' and a letter will be written for your file and to the Office of Student Conduct.

$\tan^{-1}\frac{1}{2} = 26.6°$	$\tan^{-1} 1 = 45°$
$\tan^{-1}\frac{1}{3} = 18.4°$	$\tan^{-1}\frac{1}{4} = 14°$
$\tan^{-1}\sqrt{3} = 60°$	$\tan^{-1}\frac{1}{\sqrt{3}} = 30°$
$\sin 30° = \frac{1}{2}$	$\cos 60° = \frac{\sqrt{3}}{2}$

$20\log_{10} 1 = 0dB$	$20\log_{10} 2 = 6dB$
$20\log_{10}\sqrt{2} = 3dB$	$20\log_{10}\frac{1}{2} = -6dB$
$20\log_{10} 5 = 20db - 6dB = 14dB$	$20\log_{10}\sqrt{10} = 10$ dB
$1/e \approx 0.37$	$1/e^2 \approx 0.14$
$1/e^3 \approx 0.05$	$\sqrt{10} \approx 3.16$

Key.

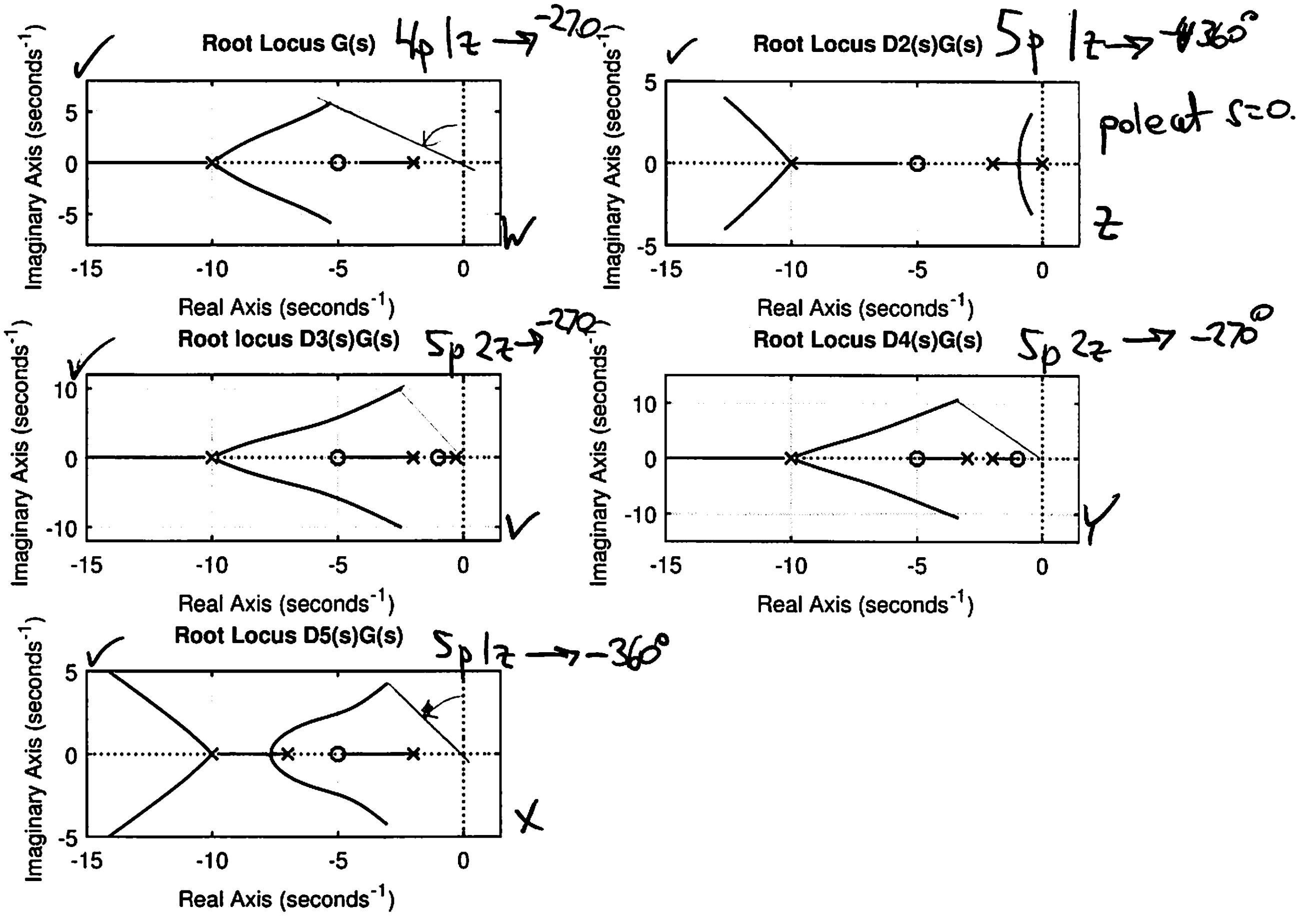
Problem 1 (14 pts)



You are given the open-loop plant:

$$G(s) = \frac{s+5}{(s+2)(s+10)^3}.$$

For the above system, the partial root locus is shown for 5 different controller/plant combinations, $G(s), D_2(s)G(s), ..., D_5(s)G(s)$. (Note: the root locus shows open-loop pole locations for $D(s)G(s)$, and closed-loop poles for $\frac{DG}{1+DG}$ are at end points of branches).



[5 pts] a) For each set of open-loop poles and zeros given above, choose the best corresponding open-loop Bode plot V,W,X,Y, or Z from the next page:

(i) $G(s)$: Bode Plot W

(ii) $D_2(s)G(s)$: Bode plot Z

(iii) $D_3(s)G(s)$: Bode plot V

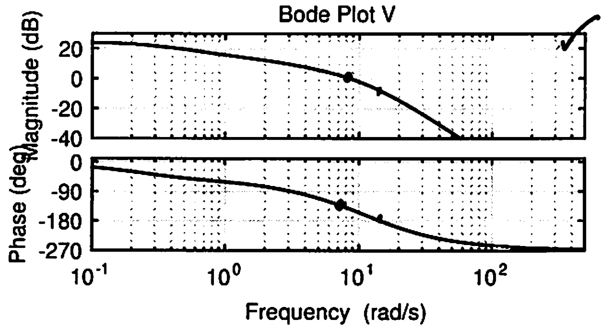
(iv) $D_4(s)G(s)$: Bode Plot Y

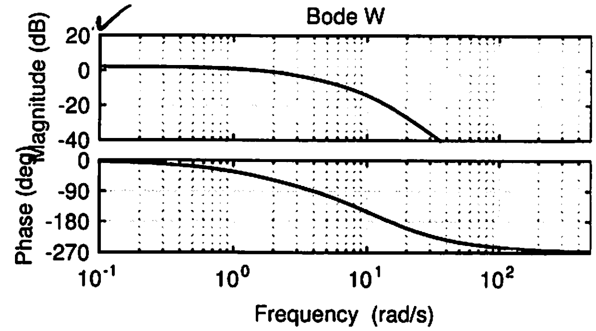
(v) $D_5(s)G(s)$: Bode Plot X

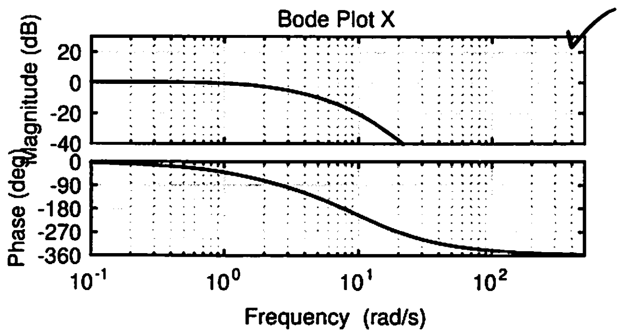
Key

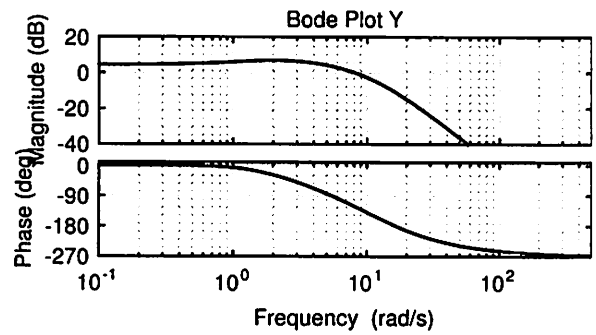
Problem 1, cont.

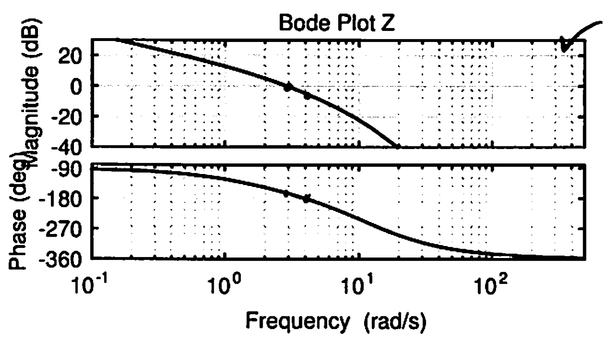
The open-loop Bode plots for 5 different controller/plant combinations, $D_1(s)G(s), \ldots, D_5(s)G(s)$ are shown below.











[4 pts] b) For the listed Bode plots, estimate the phase and gain margin:

(i) Bode plot V: phase margin 40° (degrees) at $\omega =$ 8
Bode plot V: gain margin 10 dB at $\omega =$ 13

(ii) Bode plot Z: phase margin 20 (degrees) at $\omega =$ ~~10~~ 3
Bode plot Z: gain margin 5 dB at $\omega =$ ~~10~~ 4

Key

Problem 1, cont.

[5 pts] c) For each closed loop controller/plant with root locus as given in part a), choose the best corresponding closed-loop step response (A-E)

(i) $G(s)$: step response E

(ii) $D_2(s)G(s)$: step response A

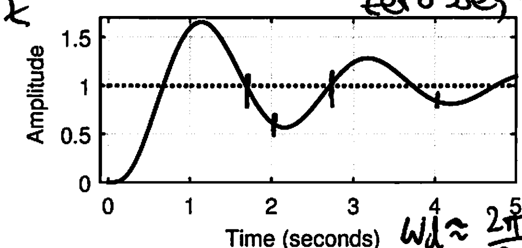
(iii) $D_3(s)G(s)$: step response D (=V)

(iv) $D_4(s)G(s)$: step response C

(v) $D_5(s)G(s)$: step response B

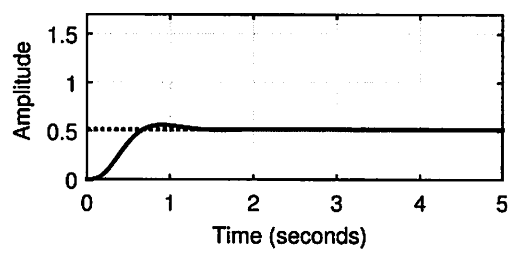
plot 2

Step A zero sse, underdamped.

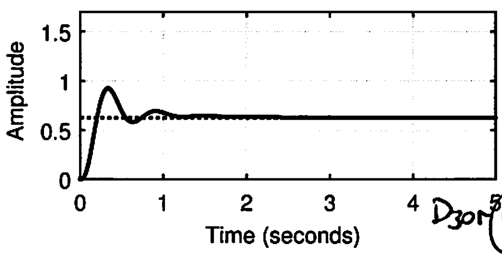


$\omega_d \approx \frac{2\pi}{2} = \pi$

Step B G or D5G



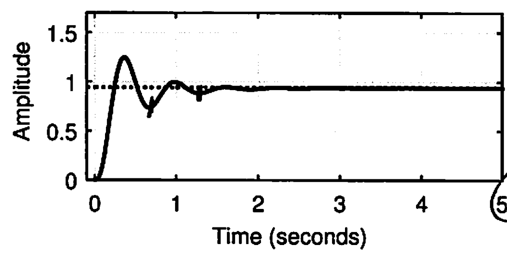
Step C



D3 or D4

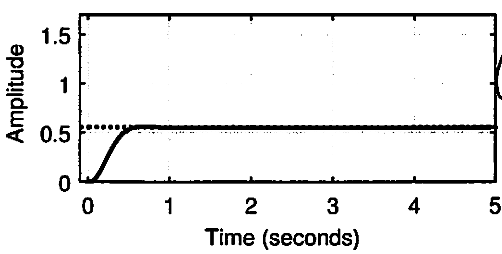
Step D

$\omega_d \approx \frac{2\pi}{0.5 sec} \approx 12$



D3 or D4

Step E

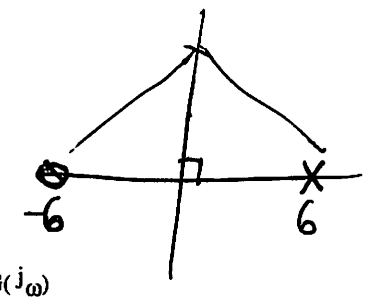


G or D5G

D4 has greater ζ than D3.

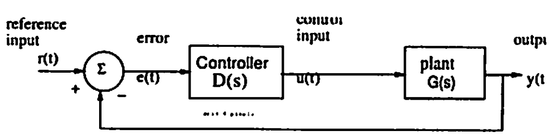
Key

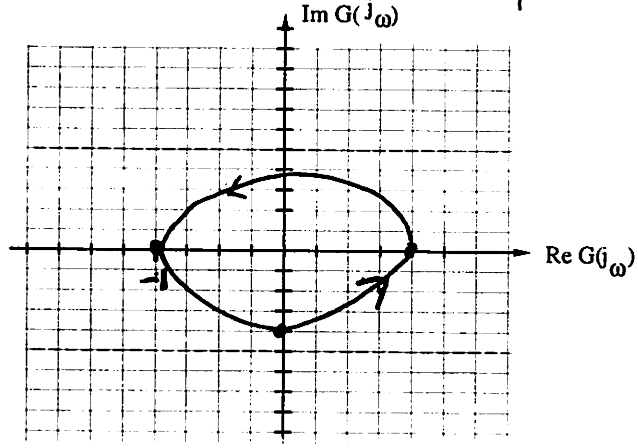
Problem 2 (14 pts)



[4 pts] a. You are given the open loop plant: $G(s) = k\frac{s+6}{s-6}$, with $D(s) = 1$. Sketch Nyquist plot for $G(s)$ with $k = 1$, showing clearly any encirclements.

ω	$\|G\|$	$\angle G$
0	1	-180°
6	1	$45^\circ - 135^\circ = -90^\circ$
$6\sqrt{3}$	1	$60^\circ - 120^\circ = -60^\circ$
∞	1	$90^\circ - 90^\circ = 0$





[2 pts] b. Find the bounds on k for the system with unity feedback to be stable.

$K > 1$ gives 1 CCW encirclement of -1

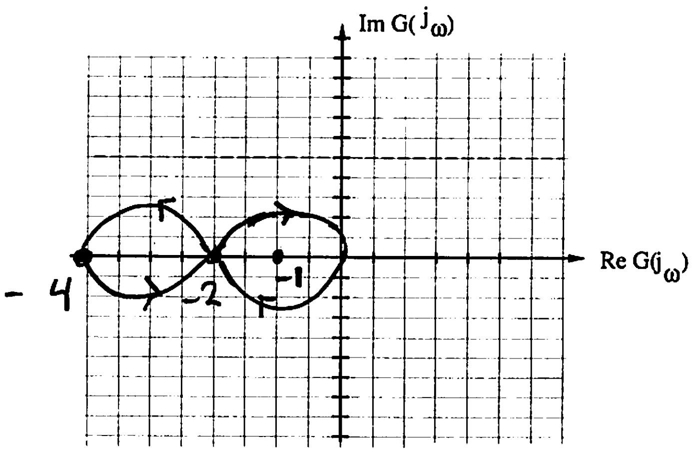
$Z = P - N$ $P = 1$ O.L. RHP

$Z = 0$. $N = 1$ CCW

[4 pts] c. You are given the open loop plant $G(s) = k\frac{(s-6)(s-4)}{(s+6)(s-1)}$. $|G| = \left|\frac{s-6}{s+6}\right| \cdot \left|\frac{s-4}{s-1}\right|$

Sketch Nyquist plot for $G(s)$ with $k = 1$, showing clearly any encirclements. Hint: phase of $G(j\omega = 2)$ is $+180^\circ$.

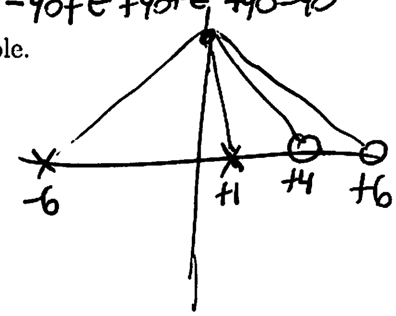
ω	$\|G\|$	$\angle G$
0	4	$180 + 180 - 180 = 180$
1	$\frac{\sqrt{17}}{\sqrt{2}} \approx 3$	
2	$\frac{\sqrt{20}}{\sqrt{5}} = 2$	180
6	$\sqrt{\frac{52}{37}} \approx \sqrt{1.4} \approx 1.2$,	$\angle \approx -45^\circ + 135^\circ - 110^\circ + 120^\circ \sim 100^\circ$
∞	0	$-90 + \epsilon + 90 + \epsilon + 90 - 90$



[4 pts] d. Find the bounds on k for the system with unity feedback to be stable.

$P = 1$ So need $N = 1$

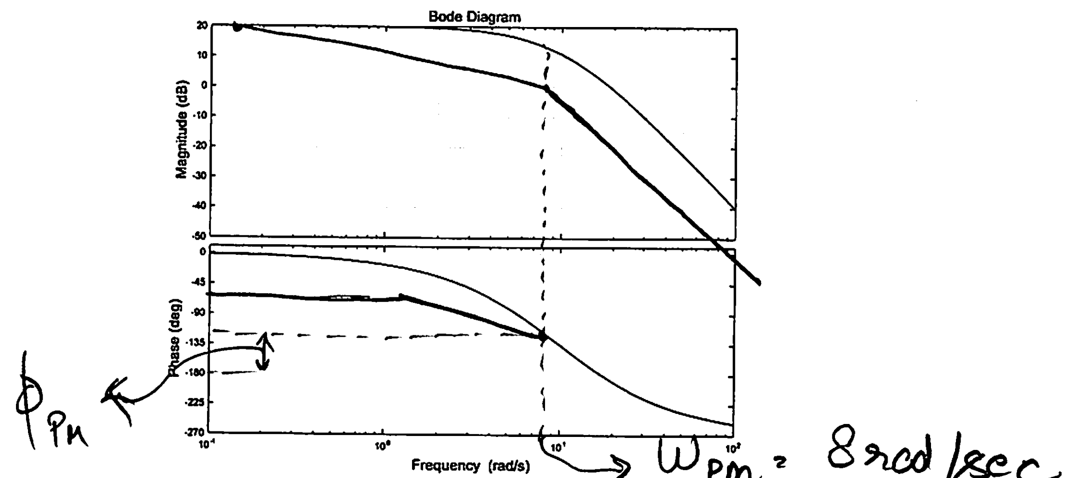
for $N = 1$, need $\frac{1}{4} < K < \frac{1}{2}$



Solution (key)

Problem 3 (16 pts)

The open-loop system is given by $G(s) = \frac{10^4}{(s+10)^3}$, and Bode plot for $G(s)$ is here (Fig. 3.1):

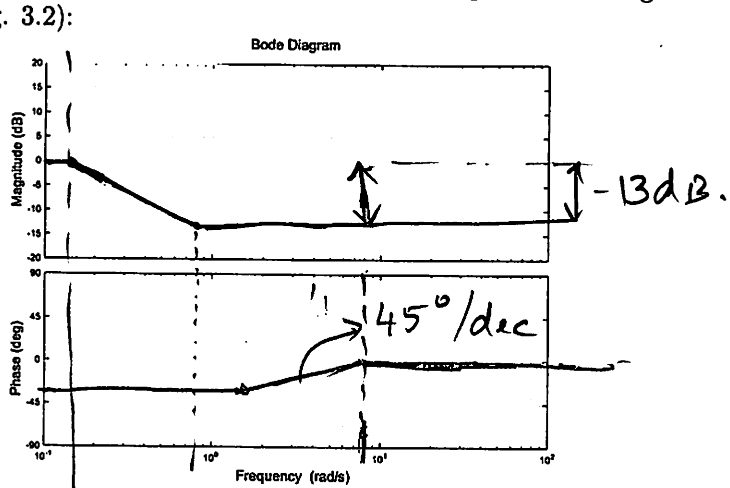


A lag controller $D(s) = k\frac{s+\alpha}{s+\beta}$ is to be designed such that the unity gain feedback system with openloop transfer function $D(s)G(s)$ has the same steady state error as with OLTF $G(s)$ and has a nominal (asymptotic approximation) phase margin $\phi_m = 55°$ at $\omega = 8$ rad s^{-1}. Note $20 \log |G(j\omega = j8)| = 13$ dB.

[6 pts] a. Determine gain, zero, and pole location for the lag network $D(s)$:

gain $k = \left(\frac{\beta}{\alpha}\right) = \frac{2}{13}$ zero $\alpha = -(0.8)$ pole $\beta = -(0.12)$

[4 pts] b. Sketch the asymptotic Bode plot for the lag network $D(s)$ alone on the plot below (Fig. 3.2):



[4 pts] c. Sketch the asymptotic Bode plot for the combined lag network and plant $D(s)G(s)$ on the plot (3.1) above.

[2 pts] d. Mark the phase and phase margin frequency on the plot of $D(s)G(s)$ (Fig. 3.1). Explain briefly (1 sentence) how does the actual phase margin compare to the asymptotic predction?

Actual Phase margin will be lesser than the prediction.

due to asymptotic approximation.

Key.

Problem 4 (8 pts)

You are given the following plant

$$\dot{\mathbf{x}} = A\mathbf{x} + Bu, \qquad y = C\mathbf{x}$$

where A is $N \times N$, u is scalar, B is $N \times 1$, C is $1 \times N$, and $\mathbf{x}$ is $N \times 1$. The system is observable and controllable.

[2 pts] a. Consider a controller $u = r - K\mathbf{x}$ where r is a reference input, and K is $1 \times N$.

A

Determine the transfer function $\frac{Y(s)}{R(s)} =$ $C[sI-A+BK]^{-1}B$

$\dot{x} = (A-BK)x + Br$

$(sI-A+BK)X(s) = BR(s)$

$X(s) = [sI-A+BK]^{-1}BR(s)$

$Y(s) = C[sI-A+BK]^{-1}BR(s)$

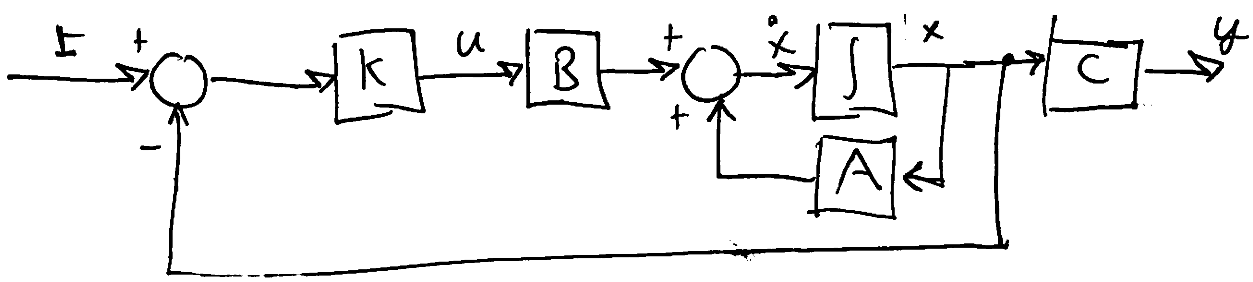
[2 pts] b. Consider a controller $u = K(\underline{r} - \mathbf{x})$ where r is a reference input, and K is $1 \times N$.

B

Determine the transfer function $\frac{Y(s)}{R(s)} =$ $C[sI-A+BK]^{-1}BK$

$\dot{x} = (A-BK)x + BKr$

[2 pts] c. Draw a block diagram of the controlled system in part b using integrators, summing junctions, and scaling functions.



[3 pts] d. If the same K is used in part a. and b. above, briefly explain any difference in transfer function or behavior:

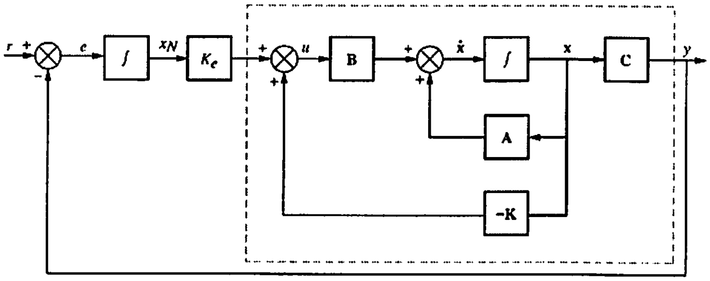
Case A and B have same eigenvalues, hence same stability and dynamic response.

In Case B, input r is vector, not scalar.

In Case B state $\underline{x}$ and output y are scaled by K compared to case A.

Key.

Problem 5. (13 pts)
Consider the following control system:



$x_N = \int e$

$\dot{x}_N = e = r - y$
$= r - Cx$

$\dot{x} = Ax + B(K_e x_N - Kx)$

[3 pts] a. Write the state and output equations for the system, in terms of A, B, C, K, K_e.

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{x}_N \end{bmatrix} = \underbrace{\begin{bmatrix} A-BK & BK_e \\ -C & 0 \end{bmatrix}}_{\bar{A}} \begin{bmatrix} \mathbf{x} \\ x_N \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r(t), \qquad y = [\; C \quad 0 \;] \begin{bmatrix} \mathbf{x} \\ x_N \end{bmatrix} \quad (1)$$

$K = [K_1 \quad K_2]$

[6 pts] b. Given $C = [1 \quad 0]$, $\quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\quad A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$

$$A - BK = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ K_1 & K_2 \end{bmatrix}$$

find K and K_e such that the closed loop poles are at $s = -1, -2, -4$.

$K = [\; 13 \quad 5 \;]$ $\qquad K_e = $ 8
$\quad [K_1 \quad K_2]$

$$\begin{bmatrix} \dot{x} \\ \dot{x}_N \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1-K_1 & -2-K_2 & K_e \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ x_N \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r$$

$$\begin{vmatrix} \lambda & -1 & 0 \\ 1+K_1 & \lambda+2+K_2 & -K_e \\ 1 & 0 & \lambda \end{vmatrix}$$

$$= \lambda \begin{vmatrix} \lambda+2+K_2 & -K_e \\ 0 & \lambda \end{vmatrix} + \begin{vmatrix} 1+K_1 & -K_e \\ 1 & \lambda \end{vmatrix}$$

$$= \lambda^2(\lambda+2+K_2) + \lambda(1+K_1) + K_e$$

$$\lambda^3 + \lambda^2(2+K_2) + \lambda(1+K_1) + K_e = 0$$

$$(s+1)(s+2)(s+4) = (s^2+3s+2)(s+4)$$
$$= s^3 + 7s^2 + 14s + 8$$

[4 pts] c. Show, with $r(t)$ a unit step input, that $e = 0$ in steady state (with the K, K_e found above). (Hint: do not use matrix inverse.)

$e = r - Cx_{ss}$

$\dot{x}_N = 0 \Rightarrow e = 0$

steady state

$$\begin{bmatrix} \dot{x} \\ \dot{x}_N \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -14 & -7 & 8 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_N \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$0 = x_2$

$0 = -14x_1 - 7x_2 + 8x_N \qquad \Longrightarrow \qquad x_2 = 0, \; x_1 = 1 \Rightarrow y = 1$

$e = r - y = 0$ ✓

$0 = -x_1 + 1 \qquad -14x_1 + 8x_N = 0, \; x_N = \frac{7}{4}.$

8

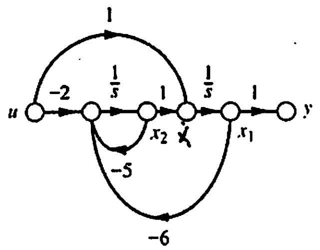
Key.

Problem 6. 13 pts

$y = x_1$

$\dot{x}_1 = x_2 + u$

$\dot{x}_2 = -6x_1 - 5x_2 - 2u$



Given the following system model:

[3 pts] a. Write the state and output equations for the system above.

$$\dot{\mathbf{x}} = A\mathbf{x} + Bu = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \end{bmatrix} u(t), \qquad y = C\mathbf{x} = [\ 1\ \ 0\ ] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

[2 pts] b. Determine if the system A, B, C is controllable and observable.

$$\mathcal{C} = [B \ \ AB] = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}, \ \det = 0, \text{ not controllable}$$

$$\mathcal{O} = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ \det = 1, \text{ observable.}$$

[2 pts] c. Provide state equations for an observer which takes as inputs $u(t), y(t)$, and provides an estimate of the state $\hat{\mathbf{x}}(t)$.

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y}) = A\hat{x} + LC(x - \hat{x}) + Bu$$
$$= (A - LC)\hat{x} + LCx + Bu$$

$$\hat{y} = C\hat{x} \qquad LC = \begin{bmatrix} l_1 & 0 \\ l_2 & 0 \end{bmatrix}$$

[6 pts] d. Find observer gain L such that the observer has closed loop poles at $s_1 = -10, s_2 = -10$.

$$(s+10)(s+10) = s^2 + 20s + 100 \ =: \text{char poly.}$$

C.L.

$$|\lambda I - (A - LC)| = \begin{vmatrix} \lambda + l_1 & -1 \\ +6 + l_2 & \lambda + 5 \end{vmatrix} = (\lambda + l_1)(\lambda + 5) + 6 + l_2$$
$$= \lambda^2 + (l_1 + 5)\lambda + 5l_1 + 6 + l_2$$

$$\Rightarrow (l_1 + 5) = 20, \ l_1 = 15$$
$$5 \cdot l_1 + 6 + l_2 = 100 \Rightarrow L = \begin{bmatrix} 15 \\ 19 \end{bmatrix}$$
$$l_2 + 6 = 25$$
$$l_2 = 19$$

Key.

Problem 7 (14 pts)

[3 pts] a. Given $G(s) = \frac{1}{s+2}$. Let $m(t)$ be the step response of $g(t)$, i.e. $M(s) = \frac{1}{s(s+2)}$. Let $x_1(t) = m(t) - m(t-T)$ where T is the sampling period. Find $X_1(z)$ the Z transform of $x_1(t)$.

$$\frac{1}{s(s+2)} = \frac{1/2}{s} + \frac{-1/2}{s+2} \longrightarrow \frac{1}{2}(1-e^{-2t})u(t) = m(t)$$

$$\frac{1}{2}u(nT) \xrightarrow{z} \frac{1}{2}\sum_{n=0}^{\infty} z^{-n} = \frac{1}{2}\frac{1}{1-z^{-1}} \quad \Big| \quad \frac{1}{2}e^{-2nT} \xrightarrow{z} \frac{1}{2}\sum_{n=0}^{\infty}(e^{-2T}z^{-1})^n = \frac{1}{2}\frac{1}{1-e^{-2T}z^{-1}}$$

$$M(z) = \frac{1}{2}\left[\frac{z}{z-1} - \frac{z}{z-e^{-2T}}\right], \quad X_1(z) = M(z) - z^{-1}M(z)$$

$$= M(z)(1-z^{-1})$$

$$= M(z)\left(\frac{z-1}{z}\right)$$

$X_1(z) = \frac{1}{2}\left[1 - \frac{z-1}{z-e^{-2T}}\right] = \frac{1}{2}\left(\frac{1-e^{-2T}}{z-e^{-2T}}\right)$

[3 pts] b. Given $\dot{x}_2(t) = -2x_2(t) + u(t)$. Find the discrete time equivalent system using zero-order hold for input $u(t)$ and sampling period T: $x_2((k+1)T) = Gx_2(kT) + Hu(kT)$.

$$G = e^{-2T} \qquad H = \int_0^T e^{-2\lambda}\,d\lambda = -\frac{1}{2}e^{-2\lambda}\Big|_0^T = \frac{1}{2}(1-e^{-2T})$$

$$= e^{AT}$$

$G =$ e^{-2T} $\qquad H =$ $\frac{1}{2}(1-e^{-2T})$

[2 pts] c. Find the $\frac{X_2(z)}{U(z)}$ the discrete time transfer function from input u to state x_2 using the state-space form. assume zero init. cond.

$$zX_2(z) = GX_2(z) + HU(z)$$

$$(zI - G)X_2(z) = HU(z)$$

$$X_2(z) = (zI-G)^{-1}HU(z)$$

$$(z - e^{-2T})^{-1}\frac{1}{2}(1-e^{-2T})$$

$$= \frac{1}{2}\frac{1-e^{-2T}}{z-e^{-2T}}$$

$\frac{X_2(z)}{U(z)}$ $\left(\frac{1}{2}\right)\frac{1-e^{-2T}}{z-e^{-2T}}$

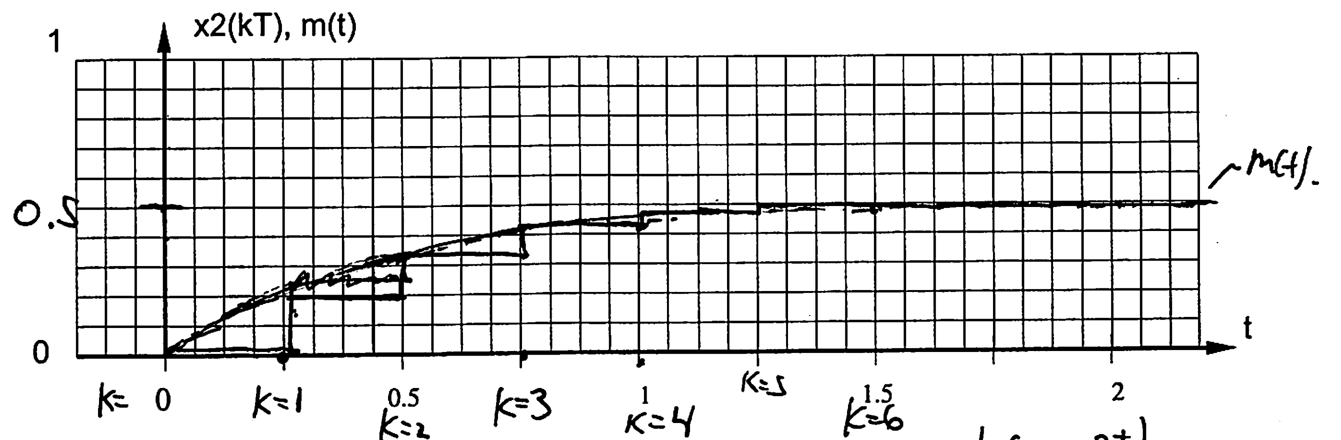
Problem 7, cont.

[2 pts] d. Does $\frac{X_2(z)}{U(z)} = X_1(z)$? Why or why not? Yes.

in part a, $x_1(t)$ is response to zero order hold.

$* \, g(t)$. Part b also uses zero order hold over T duration

[4 pts] e. With zero initial conditions (ZSR), $T = 0.25$, and a unit step input for $x_2(kT)$, sketch $m(t)$ and $x_2(kT)$ on the plot below in the interval shown:



$$m(t) = \frac{1}{2}\left(1 - e^{-2t}\right)u(t).$$

$$x_2[k+1] = e^{-0.5}x_2[k] + \frac{1}{2}\left(1 - e^{-0.5}\right)(1)$$

t		$\frac{1}{2}(1-e^{-2t})$	k
0	$e^{0} = 1$	0	0
$\frac{1}{2}$	$e^{-1} = .37$	$\frac{1}{2}(1-.37) \approx .32$	2
1	$e^{-2} = .14$	$\frac{1}{2}(1-.14) \approx .43$	4
$\frac{3}{2}$	$e^{-3} = .05$	$\frac{1}{2}(1-.05) \approx .48$	6

k	$x_2[k]$
0	0
1	$\frac{1}{2}(1-e^{-.5}) \approx \frac{1}{2}(1-.5) \approx .25$

Key.

Problem 8 Short Answers (8 pts)

[4 pts] a. Given the discrete time system below, find $\lim_{k\to\infty} x(k)$ for a unit step input $u(k) = 1$.

$$x(k+1) = \begin{bmatrix} 1/6 & 1/2 \\ 2/3 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

check stability.

$$\begin{vmatrix} \lambda - 1/6 & -1/2 \\ -2/3 & \lambda \end{vmatrix} = \lambda^2 - \frac{\lambda}{6} - \frac{2}{6} = \left(\lambda - \frac{2}{3}\right)\left(\lambda + \frac{21}{2}\right)$$

Stable $\Rightarrow$ F.V.T.

$x(k+1) = x(k)$.

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} 1/6 & 1/2 \\ 2/3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$x_1 = \frac{x_1}{6} + \frac{x_2}{2}$, $\frac{5}{3}x_1 = x_2$

$x_2 = \frac{2x_1}{3} + 1$ $\quad \frac{5}{3}x_1 = \frac{2}{3}x_1 + 1 \Rightarrow x_1 = 1$

$x_2 = \frac{5}{3}$

$\lim_{k\to\infty} x(k) = \begin{bmatrix} 1 \\ 5/3 \end{bmatrix}$.

[4 pts] b. Given $\dot{x}(t) = -10x(t) + u(t)$. The discrete time equivalent system using zero-order hold for input $u(t)$ and sampling period T is of the form $x((k+1)T) = Gx(kT) + Hu(kT)$. The discrete time system has a state feedback controller $u(kT) = r(kT) - 20x(kT)$ applied.

i. Find the eigenvalue for the closed loop system: $3e^{-10T} - 2$

ii. Find the largest value of T for which the system will be stable. (May be left in terms of ln.) : $T <$ ____

$$G = e^{aT}, \quad H = \int_0^T e^{a\lambda} B \, d\lambda = \int_0^T e^{-10\lambda} d\lambda = \left.\frac{e^{-10\lambda}}{-10}\right|_0^T = \frac{1}{10}\left(1 - e^{-10T}\right)$$

$$\begin{aligned} x(k+1) &= Gx(k) + H(r - 20x) \\ &= [G - H \cdot 20]x + Hr(k) \\ &= \left(e^{-10T} - 2 + 2e^{-10T}\right)x \\ &= 3e^{-10T} - 2 > -1 \end{aligned}$$

$3e^{-10T} > 1$

$e^{-10T} > 1/3$

$-10T > \ln 1/3$

$T < \frac{\ln 3}{10}.$