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Math54 Midterm II, Fall 2016

This is a closed book exam. Everyone is allowed a one-page cheat-sheet but no calculators. You need to justify every one of your answers unless you are asked not to do so. Completely correct answers given without justification will receive little credit. Problems are not necessarily ordered according to difficulties.

Problem	Maximum Score	Your Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	

Your Name: Richardson

Your GSI: Eric Hallman

Your SID: 26077212

1. Let $A \in \mathbf{R}^{n \times n}$ be an invertible matrix. Show that the columns of A are linearly independent.

A is an invertible matrix

Define W as all of $\mathbb{R}^n$

- $\dim W = n$

  $\therefore$ any basis for $\mathbb{R}^n$ must have n linearly independent vectors in $\mathbb{R}^n$

- By invertible matrix theorem the columns of A span $\mathbb{R}^n$

  thus the columns of A form a basis for $\mathbb{R}^n$ b/c there are n columns and all of the vectors are in $\mathbb{R}^n$

- Since the columns of A form a basis for $\mathbb{R}^n$ They must be linearly independent in order to do so.

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2. Let $\mathcal{A} = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ and $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be bases for a vector space V, and suppose

$$\mathbf{u}_1 = 4\mathbf{v}_1 - \mathbf{v}_2,\ \mathbf{u}_2 = -\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3,\ \mathbf{u}_3 = \mathbf{v}_2 - 2\mathbf{v}_3.$$

(a) Find the change-of-coordinates matrix from $\mathcal{A}$ to $\mathcal{B}$.

(b) Find $[\mathbf{x}]_{\mathcal{B}}$ for $\mathbf{x} = \mathbf{u}_1 + 2\mathbf{u}_2 + 3\mathbf{u}_3$.

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$[x]_B = P_{B \leftarrow A} [x]_A$$

$$P_{B \leftarrow A} = \big[[u_1]_B \ [u_2]_B \ [u_3]_B \big]$$

$$\left[\begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} \right]$$

$$\boxed{P_{B \leftarrow A} = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & -2 \end{bmatrix}}$$

$$[x]_A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$[x]_B = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{matrix} 4 - 2 + 0 \\ -1 + 2 + 3 \\ 0 + 2 - 6 \end{matrix} = \begin{bmatrix} 2 \\ 4 \\ -4 \end{bmatrix}$$

$$\boxed{[x]_B = \begin{bmatrix} 2 \\ 4 \\ -4 \end{bmatrix}}$$

$$\begin{aligned} x = \ & 4v_1 - v_2 + 0v_3 \\ & + (-2v_1 + 2v_2 + 2v_3) \\ & + 0v_1 + 3v_2 - 6v_3 \\ \hline & 2v_1 + 4v_2 - 4v_3 \end{aligned}$$

$$[x]_B = \begin{bmatrix} 2 \\ 4 \\ -4 \end{bmatrix} \checkmark$$

3. Find a basis for the set of all vectors of the form

$$\begin{pmatrix} a-2b+c \\ a+b+c \\ a+b \\ a+b+c \end{pmatrix},$$

where a, b, c are arbitrary constants.

$$a\begin{bmatrix}1\\1\\1\\1\end{bmatrix} + b\begin{bmatrix}-2\\1\\1\\1\end{bmatrix} + c\begin{bmatrix}1\\1\\0\\1\end{bmatrix} \qquad \cancel{a\begin{bmatrix}1\\1\\1\\1\end{bmatrix}} \quad \begin{bmatrix}-2\\1\\1\\1\end{bmatrix}$$

$$\boxed{B = \left\{ \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}-2\\1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\\1\end{bmatrix} \right\}}$$

These vectors form basis for all vectors of that form.

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4. Let

$$A = \begin{pmatrix} 7 & 2 \\ -4 & 1 \end{pmatrix}.$$

(a) Diagonalize A.

(b) Find a formula for A^k for any positive integer k.

$\begin{bmatrix} -1/2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ -2 & -1 \end{bmatrix}$

$\begin{bmatrix} -1/2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 6 & 6 \\ -10 & -5 \end{bmatrix} = \begin{matrix} -3+10 & -3+5 \\ -4 & 1 \end{matrix}$ ✓

$A - I\lambda = \begin{bmatrix} 7-\lambda & 2 \\ -4 & 1-\lambda \end{bmatrix}$

Char eq. $= (7-\lambda)(1-\lambda) - (2)(-4)$

$(7-\lambda)(1-\lambda) + 8$

$7 - 7\lambda - \lambda + \lambda^2 + 8$

$8 + 7 - 8\lambda + \lambda^2$

$= 15 - 8\lambda + \lambda^2$

$\lambda^2 - 8\lambda + 15$

$= (\lambda - 3)(\lambda - 5)$

$\lambda = 3, 5$

3: $A - I3 = \begin{bmatrix} 4 & 2 \\ -4 & -2 \end{bmatrix}$

$\begin{matrix} 4 & 2 & 0 \\ -4 & -2 & 0 \end{matrix} \sim \begin{matrix} 4 & 2 & 0 \\ 0 & 0 & 0 \end{matrix} \sim \begin{matrix} 1 & 1/2 & 0 \\ 0 & 0 & 0 \end{matrix}$

$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1/2 x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1/2 \\ 1 \end{bmatrix}$

5: $A - 5I = \begin{matrix} 2 & 2 \\ -4 & -4 \end{matrix}$

$\begin{matrix} 2 & 2 & 0 \\ -4 & -4 & 0 \end{matrix} \sim \begin{matrix} 2 & 2 & 0 \\ 0 & 0 & 0 \end{matrix} \sim \begin{matrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{matrix}$

$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$A = PDP^{-1}$ $\quad P = \begin{bmatrix} -1/2 & -1 \\ 1 & 1 \end{bmatrix}$

$D = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}$

$\boxed{A = \begin{bmatrix} -1/2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ -2 & -1 \end{bmatrix}}$

$\boxed{A^k = \begin{bmatrix} -1/2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3^k & 0 \\ 0 & 5^k \end{bmatrix} \begin{bmatrix} 2 & 2 \\ -2 & -1 \end{bmatrix}}$

$P^{-1} = ?$

$\left[\begin{matrix} -1/2 & -1 \\ 1 & 1 \end{matrix} \middle| \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}\right] \sim \left[\begin{matrix} -1 & -2 \\ 1 & 1 \end{matrix} \middle| \begin{matrix} 2 & 0 \\ 0 & 1 \end{matrix}\right]$

$\sim \left[\begin{matrix} -1 & -2 \\ 0 & -1 \end{matrix} \middle| \begin{matrix} 2 & 0 \\ 2 & 1 \end{matrix}\right] \sim \begin{matrix} -1 & 0 & -2 & -2 \\ 0 & 1 & -2 & -1 \end{matrix}$

$P^{-1} = \begin{bmatrix} 2 & 2 \\ -2 & -1 \end{bmatrix}$

Didn't know if 1 or 2 matrices necessary

$A^k = \begin{bmatrix} -1/2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \cdot 3^k & 2(3^k) \\ -2(5^k) & -5^k \end{bmatrix}$

$= \begin{bmatrix} -(3^k) + 2(5^k) & -1(3^k) + 5^k \\ 2 \cdot 3^k + (-2(5^k)) & 2 \cdot 3^k - 5^k \end{bmatrix}$

$PP^{-1} = \begin{bmatrix} -1/2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} -1+2 & -1+1 \\ 0 & \end{bmatrix}$ ✓

if $k = 1$

$-3 + 10 = 7$ ✓ $\quad -3 + 5 = 2$ ✓

$6 + -10 = -4$ ✓ $\quad 6 - 5 = 1$ ✓

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5. For

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

(a) Let $\mathbf{q_2} = \mathbf{u}_2/\|\mathbf{u}_2\|$. Find a unit vector $\mathbf{q_1}$ that is orthogonal to $\mathbf{q_2}$ so that

$$\mathbf{span}\{\mathbf{q}_1, \mathbf{q}_2\} = \mathbf{span}\{\mathbf{u}_1, \mathbf{u}_2\}.$$

(b) Let $U = (\mathbf{u}_1,\ \mathbf{u}_2)$ and $Q = (\mathbf{q}_1,\ \mathbf{q}_2)$. Find a lower-triangular matrix L, so

$$U = Q\,L.$$

$q_2 = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 1 \\ 1 \end{bmatrix} / 4$

$$\boxed{q_1 = \begin{bmatrix} -1/2 \\ 0 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}}$$

using $Q^T U = L$

$\begin{bmatrix} -1/2 & 0 & 1/2 & 1/2 & 1/2 \\ 3/4 & 1/2 & 1/4 & 1/4 & 1/4 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & 2 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$

1 0

2 4 + 3/4

project onto u_1

$$q_1 = u_1 - \frac{u_1 \cdot u_2}{u_2 \cdot u_2} u_2$$

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 3 \\ 2 \\ 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}} u_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{3+2+3}{16} \begin{bmatrix} 3 \\ 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 3 \\ 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \begin{matrix} 3/2 \\ 1 \\ 1/2 \\ 1/2 \\ 1/2 \end{matrix} = \begin{matrix} -1/2 \\ 0 \\ 1/2 \\ 1/2 \\ 1/2 \end{matrix}$$

$\|q_1\| = \sqrt{(1/2)^2 + (1/2)^2 + (1/2)^2 + (1/2)^2} = 1$

$QL = U$

$$\begin{bmatrix} -1/2 & 3/4 \\ 0 & 1/2 \\ 1/2 & 1/4 \\ 1/2 & 1/4 \\ 1/2 & 1/4 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & 2 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\begin{matrix} -1/2 & 3/4 & 1 & 3 \\ 0 & 1/2 & 1 & 2 \\ 1/2 & 1/4 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} = \begin{matrix} -1/2 & 3/4 & 1 & 3 \\ 0 & 1/2 & 1 & 2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}$$

$$\begin{matrix} -1/2 & 3/4 & 1 & 3 \\ 0 & 1/2 & 1 & 2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} = \begin{matrix} -2(-1/2 & 3/4 & 1 & 3) \\ 2(0 & 1/2 & 1 & 2) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}$$

$$\begin{matrix} -1/2 & 3/4 & 1 & 3 \\ 0 & 1 & 2 & 4 \\ 1 & -3/2 & -2 & -6 \\ 0 & 1 & 2 & 4 \end{matrix}$$

$$\begin{matrix} 1 & -3/2 & -2 & -6 \\ 0 & 1 & 2 & 4 \end{matrix} \qquad \begin{matrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 4 \end{matrix}$$

$\frac{3}{2} 2 = 3$

$\frac{3}{2} 4 = 6$

$$\boxed{L = \begin{bmatrix} 1 & 0 \\ 2 & 4 \end{bmatrix}}$$

$$\begin{bmatrix} -1/2 & 3 \\ 0 & 2 \\ 1/2 & 1 \\ 1/2 & 1 \\ 1/2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 4 \end{bmatrix} = \begin{matrix} -1/2 + 6 \\ 4 \\ 5/2 \end{matrix}$$