Midterm #2, Physics 137A, Spring 2017. Write your responses below or on extra pages. Show your work, and take care to explain what you are doing; partial credit will be given for incomplete answers that demonstrate some conceptual understanding. Cross out or erase parts of the problem you wish the grader to ignore. **Some equations of potential relevance are given on the back page**.

Problem 1: (15 pts)

Consider a spin 1/2 particle in the spin state (written in the basis of $\hat{S}_z$ eigenstates)

$$\vec{\chi} = A \begin{pmatrix} e^{i\delta} \\ 1 \end{pmatrix} \qquad \text{(where } A \text{ and } \delta \text{ are real constants)} \tag{1}$$

1a) Determine the constant A needed to normalize $\vec{\chi}$

ANSWER: The normalization requirement is

$$\vec{\chi}^\dagger \vec{\chi} = |a|^2 + |b|^2 = 1 \tag{2}$$

where a and b are the two complex coefficients of the spinor. In this example

$$(Ae^{i\delta})(A^* e^{-i\delta}) + AA^* = 1 \rightarrow |A|^2 + |A|^2 = 1 \rightarrow \boxed{|A| = 1/\sqrt{2}} \tag{3}$$

There is an arbitrary overall complex phase that could be added to A, but since this makes no difference we set it to zero and choose A real.

1b) What is the probability that a measurement of $\hat{S}_z$ for the particle returns spin up in the z-direction?

ANSWER: This and the next part of this problem are a specific case of Problem 2 of HW#6.The probability of finding spin up in z-direction is

$$P(\uparrow_z) = |\langle \uparrow_z | \chi \rangle|^2 \tag{4}$$

Where the state represented by $\vec{\chi}$ written in its abstract form is

$$|\chi\rangle = \frac{e^{i\delta}}{\sqrt{2}} |\uparrow_z\rangle + \frac{1}{\sqrt{2}} |\downarrow_z\rangle \tag{5}$$

The orthonormality of the basis vectors means

$$\langle \uparrow_z | \chi \rangle = \frac{e^{i\delta}}{\sqrt{2}} \tag{6}$$

$$P(\uparrow_z) = |\langle \uparrow_z | \chi \rangle|^2 = \left(\frac{e^{-i\delta}}{\sqrt{2}} \right) \left(\frac{e^{i\delta}}{\sqrt{2}} \right) \boxed{= \frac{1}{2}} \tag{7}$$

Since $\vec{\chi}$ is the representation in the S_z basis, we could have just written the answer down immediately as the absolute value squared of the first component of $\vec{\chi}$

1c) What is the probability that a measurement of $\hat{S}_x$ for the particle returns spin up in the x-direction?

ANSWER: The probability of finding spin up in x-direction is

$$P(\uparrow_x) = |\langle \uparrow_x | \chi \rangle|^2 \tag{8}$$

Where $|\uparrow_x\rangle$ can be written in terms of the S_z basis vectors as

$$|\uparrow_x\rangle = \frac{1}{\sqrt{2}} |\uparrow_z\rangle + \frac{1}{\sqrt{2}} |\downarrow_z\rangle \leftrightarrow \vec{\chi}_{\uparrow_x} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{9}$$

(If this solution for the spin up eigenstate of S_x was not known, it can be determined in the ordinary way by solving for the eigenvectors of the S_x matrix given on the equation sheet). Then

$$\langle \uparrow_x | \chi \rangle = \left(\frac{1}{\sqrt{2}} \langle \uparrow_z | + \frac{1}{\sqrt{2}} \langle \downarrow_z | \right) \left(\frac{e^{i\delta}}{\sqrt{2}} | \uparrow_z \rangle + \frac{1}{\sqrt{2}} | \downarrow_z \rangle \right) = \frac{e^{i\delta}}{2} + \frac{1}{2} \tag{10}$$

And so

$$P(\uparrow_x) = |\, \langle \uparrow_x | \chi \rangle \,|^2 = \left(\frac{e^{-i\delta}}{2} + \frac{1}{2} \right) \left(\frac{e^{i\delta}}{2} + \frac{1}{2} \right) = \frac{1}{4} \left(1 + e^{i\delta} + e^{-i\delta} + 1 \right) = \frac{1}{4} \left(2 + 2\cos\delta \right) \tag{11}$$

So

$$\boxed{P(\uparrow_x) = \frac{1}{2} \left(1 + \cos\delta \right)} \tag{12}$$

as a sanity check, we look at obvious the limits. For $\delta = 0$ we get $P = 1$, which makes sense since then the state $\vec{x}$ is just the spin-up in x eigenstate. For $\delta = \pi$ we get $P = 0$, which makes sense since then the state $|x\rangle$ is just the spin-down in x eigenstate.

An alternative (but mathematically equivalent) approach to solving this problem is to write the spinor $\vec{\chi}$ in the S_x basis by applying a similarity transformation

$$\vec{\chi}_{Sx} = \mathbf{S}\vec{\chi} \qquad \text{where} \;\; \mathbf{S} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{13}$$

The similarity matrix $\mathbf{S}$ is just the one made up of the eigenvectors of S_x written in the S_z basis (as done in HW#5). The spinor in the S_x basis is then

$$\vec{\chi}_{Sx} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\delta} \\ 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} e^{i\delta} + 1 \\ e^{i\delta} - 1 \end{pmatrix} \tag{14}$$

And since the spinor is now represented in the S_z basis, the answer is just the absolute value squared of the first component.

1d) For a state of the form of Eq. 1, use the non-commutation of $\hat{S}_z$ and $\hat{S}_x$ to derive an uncertainty relation written in the form

$$\sigma_{Sz}\sigma_{Sx} \geq f(\delta) \tag{15}$$

where $f(\delta)$ is a function of δ and fundamental constants. The quantities σ_{Sz} and σ_{Sx} here are the uncertainties (i.e., standard deviations) in measurements of $\hat{S}_z$ and $\hat{S}_x$ (they're not the Pauli spin matrices!)

ANSWER: The generalized uncertainty relation (given on the back equation sheet) is

$$\sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2i} \langle [\hat{A}, \hat{B}] \rangle \right)^2 \tag{16}$$

In this case, the operators are $\hat{S}_z$ and $\hat{S}_x$. We know the commutation relations for the spin operators are

$$[\hat{S}_z, \hat{S}_z] = i\hbar \hat{S}_y \tag{17}$$

(or this relation could be directly calculated from the spin matrices on the equation sheet). The uncertainty relation is then

$$\sigma_{Sz}^2 \sigma_{Sx}^2 \geq \left(\frac{1}{2i} i\hbar \langle \hat{S}_y \rangle \right)^2 \tag{18}$$

We need the expectation value of $\hat{S}_y$ for this state, which can be calculated using the spin matrix

$$\langle \hat{S}_y \rangle = \vec{\chi}^\dagger \mathbf{S_y} \vec{\chi} = \begin{pmatrix} \frac{1}{\sqrt{2}} e^{-i\delta} & \frac{1}{\sqrt{2}} \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\delta} \\ 1 \end{pmatrix} \tag{19}$$

$$\langle \hat{S}_y \rangle = \frac{\hbar}{4} \begin{pmatrix} e^{-i\delta} & 1 \end{pmatrix} \begin{pmatrix} -i \\ ie^{i\delta} \end{pmatrix} = \frac{\hbar}{4} \left(-ie^{-i\delta} + ie^{i\delta} \right) = -\frac{\hbar}{4}(2\sin\delta) = -\frac{\hbar}{2}\sin\delta \tag{20}$$

So plugging this in above gives

$$\sigma_{Sz}^2 \sigma_{Sx}^2 \geq \left(-\frac{\hbar}{2}\frac{\hbar}{2}\sin\delta \right)^2 = \frac{\hbar^4}{16}\sin^2\delta \tag{21}$$

Taking the square root

$$\boxed{\sigma_{Sz}\sigma_{Sx} \geq \frac{\hbar^2}{4}|\sin\delta|} \tag{22}$$

Problem 2: (12 pts)

Let us denote the normalized energy eigenstates of the infinite square well as $|1\rangle, |2\rangle, |3\rangle, ... |n\rangle$. Consider the operator

$$\hat{Q} = |1\rangle \langle 2| \tag{23}$$

2a) Show that the value[1] $\langle Q \rangle = 0$ for any of the energy eigenstates $|n\rangle$.

ANSWER: The quantity requested is

$$\langle Q \rangle = \left\langle n \middle| \hat{Q} n \right\rangle = \langle n|1\rangle \langle 2|n\rangle \tag{24}$$

By the orthogonality of the eigenstates, $\langle n|1\rangle = 0$ unless $|n\rangle = |1\rangle$, but in this case $\langle 2|n\rangle = \langle 2|1\rangle = 0$, so $\langle Q \rangle = 0$ for any state $|n\rangle$.

2b) Find the value $\langle Q \rangle$ for the state $|\alpha\rangle = \frac{1}{\sqrt{2}} i |1\rangle + \frac{1}{\sqrt{2}} |2\rangle$

ANSWER: The quantity requested

$$\langle Q \rangle = \left(\frac{1}{\sqrt{2}} (-i) \langle 1| + \frac{1}{\sqrt{2}} \langle 2| \right) |1\rangle \langle 2| \left(\frac{1}{\sqrt{2}} i |1\rangle + \frac{1}{\sqrt{2}} |2\rangle \right) \tag{25}$$

where we were careful to complex conjugate the i when taking the corresponding bra of the ket. The result is, using the orthogonality of the eigenstates

$$\langle Q \rangle = \frac{-i}{\sqrt{2}} \frac{1}{\sqrt{2}} = \frac{-i}{2} \tag{26}$$

Since $\hat{Q}$ is not Hermitian, there is no requirement that this value be real (as alluded to in the footnote).

2c) Does $\hat{Q}$ commute with the hamiltonian $\hat{H}$? Based on this result determine whether $\langle Q \rangle$ remains constant over time.

ANSWER: We write the commutator

$$[\hat{H}, \hat{Q}] = \hat{H} |1\rangle \langle 2| - |1\rangle \langle 2| \hat{H} \tag{27}$$

This can be calculated abstractly by using the fact that $|1\rangle$ and $|2\rangle$ are eigenstates of $\hat{H}$ with eigenvalues E_1 and E_2, respectively. We use the fact that $\hat{H}$ is hermitian to move it over onto the $\langle 2|$

$$[\hat{H}, \hat{Q}] = \hat{H} |1\rangle \langle 2| - |1\rangle \left\langle \hat{H}^\dagger 2 \right| = \hat{H} |1\rangle \langle 2| - |1\rangle \left\langle \hat{H} 2 \right| = E_1 |1\rangle \langle 2| - E_2 |1\rangle \langle 2| \tag{28}$$

hence

$$[\hat{H}, \hat{Q}] = (E_1 - E_2) |1\rangle \langle 2| \neq 0 \tag{29}$$

If you are uncomfortable with the moving of $\hat{H}$ into the $\langle 2|$ term, you can instead apply the commutator to a general state written as a sum of eigenstates

$$[\hat{H}, \hat{Q}] |\alpha\rangle = \left(\hat{H} |1\rangle \langle 2| - |1\rangle \langle 2| \hat{H} \right) \sum_n c_n |n\rangle \tag{30}$$

which is seen to be

$$= (E_1 |1\rangle \langle 2| - |1\rangle \langle 2| E_2) \sum_n c_n |n\rangle = (E_1 - E_2) |1\rangle \langle 2| |\alpha\rangle \tag{31}$$

So the commutator again is not zero.

[1] Note that if $\hat{Q}$ is not Hermitian, its eigenvalues are not guaranteed to be real, and so the quantity $\langle Q \rangle = \left\langle \alpha \middle| \hat{Q} \alpha \right\rangle$ for some state $|\alpha\rangle$ may not actually represent the average value of measurements of some real physical observable.

An even simpler way to do the problem would be to just look at the commutator on the specific state $|2\rangle$. Then we see

$$[\hat{H}, \hat{Q}]\,|2\rangle = \hat{H}\,|1\rangle\,\langle 2|\,|2\rangle - |1\rangle\,\langle 2|\,\hat{H}\,|2\rangle = |1\rangle\,(E_1 - E_2) \neq 0 \tag{32}$$

This is a specific counter-example to the claim that $\hat{H}$ and $\hat{Q}$ commute.

You derived in HW#7 (and also in class) that the time evolution of an expectation value is related to the expectation value of the commutator by

$$\frac{\partial}{\partial t}\langle Q\rangle = \frac{i}{\hbar}\left\langle [\hat{H}, \hat{Q}]\right\rangle \tag{33}$$

(which uses the fact that $\hat{Q}$ does not change with time). Since $\hat{H}$ and $\hat{Q}$ don't commute, we see that the $\langle Q\rangle$ is *not* a constant with time in general (unless it happens to be zero).

2d) Is the operator $\hat{Q}$ unitary? Demonstrate why or why not (there is more than one way to do so).

ANSWER: A unitary operator preserves the lengths of vectors. But we have from part 2a) that

$$\hat{Q}\,|n\rangle = |\emptyset\rangle \qquad \text{where } |\emptyset\rangle \text{ is the null vector} \tag{34}$$

for any eigenstate. The length of $\hat{Q}\,|n\rangle$ is zero, while the length of $|n\rangle$ is not (its length is 1 assuming the eigenstates are normalized) therefore $\hat{Q}$ is *not* unitary.

More directly, the Hermitian conjugate of $\hat{Q}$ is

$$\hat{Q}^\dagger = |2\rangle\,\langle 1| \tag{35}$$

And so

$$\hat{Q}^\dagger\hat{Q} = |2\rangle\,\langle 1|\,|1\rangle\,\langle 2| = |2\rangle\,\langle 2| \neq \hat{I} = \sum_n |n\rangle\,\langle n| \tag{36}$$

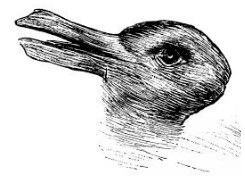
Where in the last step we conclude that the operator is not equal to the identity operator since it includes only one term in the sum giving the resolution of the identity. Since $\hat{Q}^\dagger \neq \hat{Q}^{-1}$ we see $\hat{Q}$ is not unitary.

Note that all of the parts of this problem become rather straightforward if you realize that the operator $\hat{Q}$ can be written as a matrix in the basis of energy eigenvectors (as was done in HW#7 part 4c). The action of $\hat{Q}$ is to just turn state $|2\rangle$ into state $|1\rangle$ so the matrix must be

$$\hat{Q} = \begin{pmatrix} 0 & 1 & 0 & . \\ 0 & 0 & 0 & . \\ . & . & . & . \end{pmatrix} \tag{37}$$

It is then directly clear that $\hat{Q}^\dagger \neq \hat{Q}^{-1}$ (so the operator is not unitary). One can can also calculate expectation values directly from the matrix multiplication, and can directly compute the commutation with the $\hat{H}$ as matrix multiplications (in this representation $\hat{H}$ is just diagonal with eigenvalues along the diagonal).

Problem 3: (18 pts)



Scientists have just discovered an amazing new quantum property of particles, called "animalness". When measured, a particle is observed to be either a "duck" or a "rabbit". The "animalness" is associated with a Hermitian operator $\hat{A}$ with eigenvectors $|D\rangle$ (for duck) and $|R\rangle$ (for rabbit) with eigenvalues $\lambda_D = +1$ and $\lambda_R = -1$ respectively. In the $|D\rangle, |R\rangle$ basis the $\hat{A}$ operator is represented by

$$\hat{A} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{38}$$

A particle is put into a device where the Hamiltonian (written in the $|D\rangle, |R\rangle$ basis) is

$$\hat{H} = e \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \qquad \text{(where } e \text{ is a constant)} \tag{39}$$

3a) What are the possible values of energy that can be observed?

ANSWER: We want to calculate the eigenvalues of $\hat{H}$, similar to the calculations we have done for other 2x2 matrices on the homework. The characteristic equation is

$$\det|\hat{H} - \lambda\hat{I}| = \det \begin{pmatrix} e-\lambda & e \\ e & e-\lambda \end{pmatrix} = 0 \tag{40}$$

$$(e-\lambda)(e-\lambda) - e^2 = 0 \rightarrow e^2 - 2e\lambda + \lambda^2 - e^2 = 0 \tag{41}$$

$$\lambda(\lambda - 2e) = 0 \rightarrow \boxed{\lambda_1 = 0 \text{ or } \lambda_2 = 2e} \tag{42}$$

3b) A particle with the above Hamiltonian is initially a duck (i.e., in the $|D\rangle$ state). What is the uncertainty in its energy?

ANSWER: In this basis the state $|D\rangle$ is just represented by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ so we have

$$\langle E\rangle = \begin{pmatrix} 1 & 0 \end{pmatrix} e \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = e \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = e \tag{43}$$

and

$$\langle E^2\rangle = \begin{pmatrix} 1 & 0 \end{pmatrix} e \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} e \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix} e^2 \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 2e^2 \tag{44}$$

So the uncertainty in energy is

$$\sigma_E^2 = \langle E^2\rangle - \langle E\rangle^2 = 2e^2 - e^2 = e^2 \tag{45}$$

hence $\boxed{\sigma_E = e}$.

3c) We make a measurement and determine the energy of the particle. Calculate the probability that a subsequent measurement of $\hat{A}$ will find the particle to be a duck.

ANSWER: If we measure the energy, we collapse the state to one of the energy eigenvectors. We can determine the representation of the eigenvectors in the $|D\rangle, |R\rangle$ basis in the usual way

$$e \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \end{pmatrix} \tag{46}$$

which implies

$$e(a+b) = \lambda a \tag{47}$$

for $\lambda_1 = 0$ this implies $a = -b$, while for $\lambda_2 = 2e$ we have $a = b$. Normalizing the two eigenvectors so that $|a|^2 + |b|^2 = 1$ gives

$$\vec{e}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \vec{e}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{48}$$

The probability of finding the particle to be a duck in the $|D\rangle, |R\rangle$ basis is the absolute value squared of the first component of the vector. The problem does not specify which energy is measured – and hence which eigenvector we collapse to – but we see that it doesn't matter. For both eigenvectors the probability of being in the $|D\rangle$ state is $\boxed{1/2}$

3d) We measure $\hat{A}$ and find the particle to be a duck. We then let the system evolve for a time t and measure $\hat{A}$ again. Calculate the probability that we find the particle to be a rabbit (i.e., in the $|R\rangle$ state).

ANSWER: This is similar to problems 2 and 3 on HW#6. The eigenstates of the Hamiltonian are those that evolve simply with time (by application of a complex phase). So we write the state $|D\rangle$ as a superposition of energy eigenstates

$$|D\rangle \leftrightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\vec{e}_1 + \frac{1}{\sqrt{2}}\vec{e}_2 \tag{49}$$

And since we know how the energy eigenvectors evolve we have the state at some time t as

$$\vec{\chi}(t) = \frac{1}{\sqrt{2}} e^{-iE_1 t/\hbar}\vec{e}_1 + \frac{1}{\sqrt{2}} e^{-iE_2 t/\hbar}\vec{e}_2 \tag{50}$$

so plugging in $E_1 = 0, E_2 = 2e$, and the expression for the eigenvectors gives

$$\vec{\chi}(t) = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} e^{-2iet/\hbar} \\ e^{-2iet/\hbar} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} e^{-2iet/\hbar} + 1 \\ e^{-2iet/\hbar} - 1 \end{pmatrix} \tag{51}$$

The rabbit state is represented by $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and so the probability of being observed a rabbit is the absolute value squared of the second component of $\vec{\chi}$

$$P(R) = \frac{1}{2}\left(e^{2iet/\hbar} - 1\right)\frac{1}{2}\left(e^{-2iet/\hbar} - 1\right) = \frac{1}{4}\left(1 - e^{-2iet/\hbar} - e^{2iet/\hbar} + 1\right) \tag{52}$$

$$P(R) = \frac{1}{4}\left(2 - 2\cos(2et/\hbar)\right) = \boxed{\frac{1}{2}\left(1 - \cos(2et/\hbar)\right)} \tag{53}$$

and if we care to, we see the probability of being a duck is

$$P(D) = \frac{1}{2}\left(1 + \cos(2et/\hbar)\right) \tag{54}$$

We sanity check that at $t = 0$ the probability of being a rabbit is zero, and the maximum value of $P(R)$ is 1. The sum of being a duck or a rabbit is one at all times.

Problem 4: (15 pts)

The lowering ladder operator of the harmonic oscillator $\hat{a}_-$ is not Hermitian (and so does not correspond to any observable) but we can still find eigenstates of the operator that obey

$$\hat{a}_- |\alpha\rangle = \alpha |\alpha\rangle \tag{55}$$

where $|\alpha\rangle$ is a (normalized) eigenstate of $\hat{a}_-$ and the associated eigenvalue α is some complex number. The states $|\alpha\rangle$ turn out to be quite physically interesting.

4a) Find the expectation value of $\hat{x}$ for an eigenstate $|\alpha\rangle$ in terms of α and other variables.

ANSWER: Recalling the use of ladder operators in HW#7 problems 3 and 4, we write the operator $\hat{x}$ in terms of ladder operators

$$\hat{x} = x_0 \left(\hat{a}_+ + \hat{a}_-\right) \qquad \text{where } x_0 = \sqrt{\frac{\hbar}{2m\omega_0}} \tag{56}$$

so the expectation value of $\hat{x}$ is

$$\langle x\rangle \langle\alpha|\hat{x}\alpha\rangle = x_0 \langle\alpha|(\hat{a}_+ + \hat{a}_-)\alpha\rangle = x_0 \langle\alpha|\hat{a}_+\alpha\rangle + x_0 \langle\alpha|\hat{a}_-\alpha\rangle \tag{57}$$

The second term can be evaluated since $|\alpha\rangle$ is an eigenstate of $\hat{a_-}$

$$x_0 \langle\alpha|\hat{a}_-\alpha\rangle = x_0\alpha \langle\alpha|\alpha\rangle = x_0\alpha \tag{58}$$

which uses the fact hat $|\alpha\rangle$ is normalized. To deal with the first term involving the raising operator we note that the Hermitian conjugate of $\hat{a}_+$ is $\hat{a}_+^\dagger = \hat{a}_-$. Thus the term is

$$x_0 \langle\alpha|\hat{a}_+\alpha\rangle = x_0 \left\langle \hat{a}_+^\dagger\alpha \middle| \alpha\right\rangle = x_0 \langle\hat{a}_-\alpha|\alpha\rangle = x_0\alpha^* \langle\alpha|\alpha\rangle = x_0\alpha^* \tag{59}$$

Note that since α is not necessarily real, we must complex conjugate the constant. This gives finally

$$\boxed{\langle x\rangle = x_0(\alpha^* + \alpha)} \tag{60}$$

4b) Determine the uncertainty in position σ_x for $|\alpha\rangle$.

ANSWER: To get the uncertainty we need to find $\langle x^2\rangle$. The operator is

$$\hat{x^2} = x_0(\hat{a}_+ + \hat{a}_-)x_0(\hat{a}_+ + \hat{a}_-) = x_0^2(\hat{a}_+^2 + \hat{a}_+\hat{a}_- + \hat{a}_-\hat{a}_+ + \hat{a}_-^2) \tag{61}$$

The expectation value has four terms now. Three of them are

$$\left\langle\alpha\middle|\hat{a}_-^2\alpha\right\rangle = \alpha^2 \tag{62}$$

$$\left\langle\alpha\middle|\hat{a}_+^2\alpha\right\rangle = \left\langle\hat{a}_-^2\alpha\middle|\alpha\right\rangle = \alpha^{*2} \tag{63}$$

$$\langle\alpha|\hat{a}_+\hat{a}_-\alpha\rangle = \langle\hat{a}_-\alpha|\hat{a}_-\alpha\rangle = \alpha^*\alpha \tag{64}$$

For the last term, we will need to use the commutation relation $[\hat{a}_-, \hat{a}_+] = 1$ which allows us to write $\hat{a}_-\hat{a}_+ = 1 + \hat{a}_+\hat{a}_-$

$$\langle\alpha|\hat{a}_-\hat{a}_+\alpha\rangle = \langle\alpha|\alpha\rangle + \langle\alpha|\hat{a}_+\hat{a}_-\alpha\rangle = 1 + \alpha^*\alpha \tag{65}$$

Putting it all together

$$\left\langle x^2\right\rangle = x_0^2(\alpha^2 + \alpha^{*2} + 2\alpha^*\alpha + 1) \tag{66}$$

so the uncertainty squared is

$$\sigma_x^2 = \left\langle x^2\right\rangle - \langle x\rangle^2 = x_0^2(\alpha^2 + \alpha^{*2} + 2\alpha^*\alpha + 1) - x_0^2(\alpha^2 + \alpha^{*2} + 2\alpha^*\alpha) = x_0^2 \tag{67}$$

and so $\boxed{\sigma_x = x_0}$

For the sake of time, I won't ask you to calculate the uncertainty in momentum σ_p, but if you did you would see that the $|\alpha\rangle$ are special states that have the minimum possible uncertainty $\sigma_x \sigma_p = \hbar/2$. We can express these states (as we can any state) as a superposition of the energy eigenstates of the harmonic oscillator, $|n\rangle$

$$|\alpha\rangle = \sum_{n=0}^{\infty} c_n |n\rangle \tag{68}$$

4c) Derive an expression for the coefficients c_n in terms of the coefficient c_0 of the ground state (the coefficient c_0 itself can be determined using the normalization condition, which you needn't bother to do).

ANSWER: We want to use the fact that $|\alpha\rangle$ is an eigenstate of the lowering operator, which means

$$\hat{a}_- \sum_{n=0}^{\infty} c_n |n\rangle = \alpha \sum_{n=0}^{\infty} c_n |n\rangle \tag{69}$$

It may be easiest to see this by writing out terms explicitly

$$|\alpha\rangle = c_0 |0\rangle + c_1 |1\rangle + c_2 |2\rangle + c_3 |3\rangle + ... \tag{70}$$

applying the lowering operator (its action is given on the back equation sheet) we find

$$\hat{a}_- |\alpha\rangle = c_1 |0\rangle + c_2\sqrt{2} |1\rangle + c_3\sqrt{3} |2\rangle + ... \tag{71}$$

then the first equation is

$$c_1 |0\rangle + c_2\sqrt{2} |1\rangle + c_3\sqrt{3} |2\rangle + ... = \alpha(c_0 |0\rangle + c_1 |1\rangle + c_2 |2\rangle + c_3 |3\rangle ...) \tag{72}$$

Now to pick out the coefficients, we just apply the appropriate bra. For example, applying $\langle 0|$ to each side gives

$$c_1 = \alpha c_0 \tag{73}$$

and applying $\langle 1|$ to each side gives

$$c_2\sqrt{2} = \alpha c_1 \rightarrow c_2 = \frac{\alpha}{\sqrt{2}} c_1 = \frac{\alpha^2}{\sqrt{2}} c_0 \tag{74}$$

and applying $\langle 2|$ to each side gives

$$c_3\sqrt{3} = \alpha c_2 \rightarrow c_3 = \frac{\alpha}{\sqrt{3}} c_2 = \frac{\alpha^3}{\sqrt{3}\sqrt{2}} c_0 \tag{75}$$

and we see the trend

$$\boxed{c_n = \frac{\alpha^n}{\sqrt{n!}} c_0} \tag{76}$$

To do this more generally, we can write the action of the lowering operator

$$\hat{a}_- |\alpha\rangle = \sum_{n=0}^{\infty} c_n \hat{a}_- |n\rangle = \sum_{n=1}^{\infty} c_n \sqrt{n} |n-1\rangle \tag{77}$$

and since this is an eigenstate, $\hat{a}_- |\alpha\rangle = \alpha |\alpha\rangle$ so

$$\alpha \sum_{n=0}^{\infty} c_n |n\rangle = \sum_{n=1}^{\infty} c_n \sqrt{n} |n-1\rangle \tag{78}$$

Applying $\langle n|$ to both sides to pick off the coefficients gives

$$\alpha c_n = c_{n+1}\sqrt{n+1} \tag{79}$$

so

$$c_{n+1} = \frac{\alpha}{\sqrt{n+1}} c_n \tag{80}$$

which is the pattern above.

Equations and Extra Work Space

$$i\hbar\frac{\partial}{\partial t}\left|\psi\right\rangle = \hat{H}\left|\psi\right\rangle \qquad \text{time dependent Schrodinger equation} \tag{81}$$

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega_0^2\hat{x}^2 \qquad \text{Harmonic oscillator Hamiltonian} \tag{82}$$

Harmonic oscillator ladder operators

$$\hat{a}_+ = \frac{1}{\sqrt{2m\hbar\omega_0}}\left[m\omega_0\hat{x} - i\hat{p}\right] \qquad \hat{a}_+\left|n\right\rangle = \sqrt{n+1}\left|n+1\right\rangle \tag{83}$$

$$\hat{a}_- = \frac{1}{\sqrt{2m\hbar\omega_0}}\left[m\omega_0\hat{x} + i\hat{p}\right] \qquad \hat{a}_-\left|n\right\rangle = \sqrt{n}\left|n-1\right\rangle \tag{84}$$

Commutation relations

$$[\hat{x},\hat{p}] = i\hbar \qquad [\hat{a}_-,\hat{a}_+] = 1 \tag{85}$$

Spin 1/2 operator matrices written in $\hat{S}_z$ basis

$$\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \hat{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{86}$$

Expectation value and uncertainty

$$\langle Q\rangle = \left\langle \psi \middle| \hat{Q}\psi \right\rangle \qquad \sigma_Q^2 = \left\langle Q^2\right\rangle - \langle Q\rangle^2 \qquad \sigma_A^2\sigma_B^2 \geq \left(\frac{1}{2i}\langle[\hat{A},\hat{B}]\rangle\right)^2 \tag{87}$$