Midterm #1, Physics 137A, Spring 2017. Write your responses below, on the back, or on the extra last page. Some integrals (not all needed) are provided on the last page. Show your work, and take care to explain what you are doing; partial credit will be given for incomplete answers that demonstrate some conceptual understanding. Cross out or erase parts of the problem you wish the grader to ignore.

Problem 1: (15 pts)

A particle is in a state with a spatial wavefunction given by

$$\psi(x) = Ae^{ax} \quad \text{for } x < 0 \tag{1}$$

$$\psi(x) = Ae^{-ax} \quad \text{for } x \geq 0 \tag{2}$$

where A and a are real constants.

1a) Normalize the wavefunction.

Answer: Normalization is defined by

$$\int_{-\infty}^{\infty} |\psi(x)|^2 dx = 1 \tag{3}$$

This breaks up into two integrals

$$\int_{-\infty}^{\infty} |\psi(x)|^2 dx = \int_{-\infty}^{0} |A|^2 e^{2ax} dx + \int_{0}^{\infty} |A|^2 e^{-2ax} dx \tag{4}$$

$$= \frac{|A|^2}{2a} e^{2ax}\Big|_{-\infty}^{0} + \frac{|A|^2}{-2a} e^{2ax}\Big|_{0}^{\infty} \tag{5}$$

$$= \frac{|A|^2}{2a}[1-0] - \frac{|A|^2}{2a}[0-1] = \frac{|A|^2}{a} \tag{6}$$

so the normalization conditions requires

$$|A|^2 = a \rightarrow \boxed{A = \sqrt{a}} \tag{7}$$

We could add any complex phase and write $A = \sqrt{a}e^{i\delta}$ and still satisfy the normalization requirement. However the overall phase has no physical meaning so we might as well choose $\delta = 0$.

1b) Determine the probability that a measurement finds the particle between $x = 0$ and $x = L$.

Answer: The probability is calculated by

$$P = \int_{0}^{L} |\psi(x)|^2 dx = \int_{0}^{L} ae^{-2ax} dx = \frac{a}{-2a} e^{-2ax}\Big|_{0}^{L} \tag{8}$$

$$P = \frac{-1}{2}\left[e^{-2aL} - 1\right] \rightarrow \boxed{P = \frac{1}{2}\left[1 - e^{-2aL}\right]} \tag{9}$$

and we sanity check that $P \geq 0$ and real as it should be.

1c) What is the expectation value of position, $\langle x \rangle$, for this particle? You don't have to do an explicit calculation if you can write down the answer and justify it.

Answer: This function is symmetric about x, and so we can say by symmetry that the integration of

$$\langle x \rangle = \int_{-\infty}^{\infty} |\psi(x)|^2 x dx = 0 \tag{10}$$

since x is odd. If we want to do the actual integral we will have to use the integral formulae given at the end of the exam, and will find the same answer.

1d) Roughly estimate the uncertainty in momentum σ_p for this wavefunction. You don't have to do a calculation of anything (and don't worry about getting numerical factors of 2 or so right) simply justify why σ_p should be roughly equal to or greater than your estimate.

Answer: The exponential function e^{-ax} falls off on the length scale $1/a$ – that is, at $x = 1/a$, the function has decreased by a factor of $1/e$ from the peak at $x = 0$. This is more than a factor of 2, so $1/a$ provides a rough estimate of the width (and hence uncertainty) of the wavefunction (see such an estimate in HW#1 problem 3i).

Since the problem did not ask for a detailed calculation, we can roughly estimate the uncertainty in x as $\sigma_x \sim 1/a$ (where the $\sim$ means "roughly equal to"). To get the precise uncertainty σ_x, we would need to do the integrals to calculate the expectation values $\langle x^2 \rangle - \langle x \rangle^2$, but the problem says this is not required. The Heisenberg uncertainty principle says

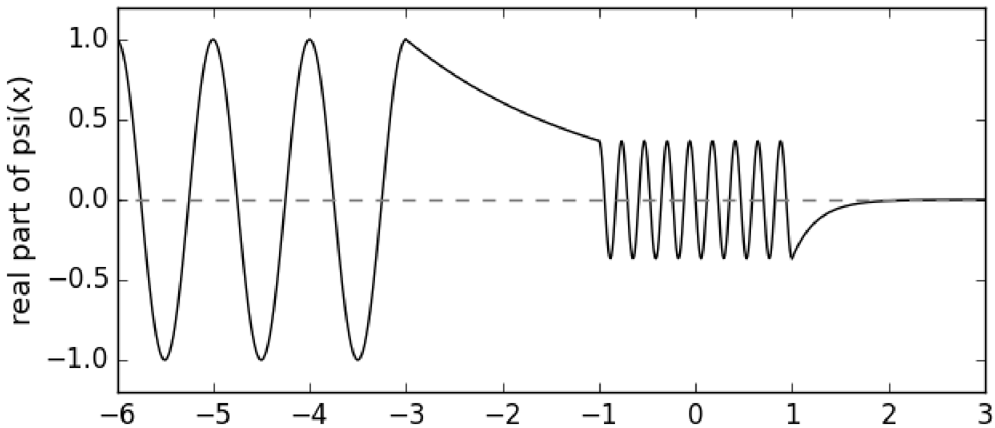
$$\sigma_p \sigma_x \geq \frac{\hbar}{2} \tag{11}$$

so this implies

$$\sigma_p \geq \frac{\hbar}{2\sigma_x} \sim \frac{\hbar a}{2} \tag{12}$$

So roughly we can say that $\boxed{\sigma_p \sim \hbar a}$.

Problem 2: (8 pts) A particle of energy E in some potential $V(x)$ has the (not normalized) wavefunction $\psi(x)$ shown in the plot below (the real part of ψ is plotted). The wavefunction extends to $x \to \pm\infty$ in a way consistent with the behavior at the edges.



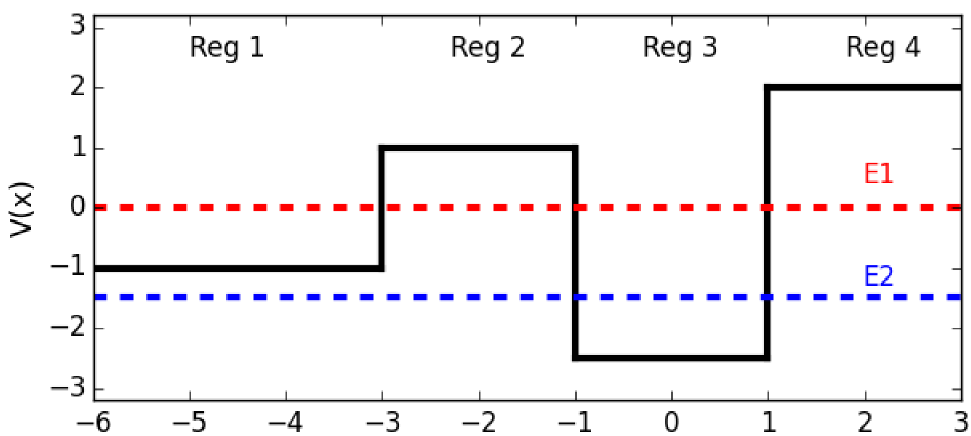
2a) Sketch approximately the potential energy function $V(x)$, and mark on it a plausible value of the energy E of the particle. Try to get the relative heights of the potential roughly correct.

Answer: We know that in a region of constant potential V_0, the wavefunction is complex exponential (oscillating) when $E > V_0$. The oscillation wavenumber is

$$k = \left[\frac{2m(E - V_0)}{\hbar^2}\right]^{1/2} \tag{13}$$

and so the wavefunction oscillates more rapidly in regions of lower V_0 (to use the classical mnemonic, the particle "moves faster" in regions of lower potential).

In regions where $E < V_0$ (the so-called classicaly forbidden region) the wavefunction behaves exponentially. This leads to the following sketch of $V(x)$



Where the line labeled E_1 gives the energy of the particle. We chose $E_1 = 0$, but the zero point of energy is arbitrary (we can always shift all energies by a constant); what matters is the relative values of energy.

The important features of this plot are 1) $E_1 > V_0$ in the regions 1 and 3, because the wavefunction is oscillating there. 2) The potential in the region 3 must be less than that of region 1 since the wavenumber of oscillation is greater there. 3) $E_1 < V_0$ in regions 2 and 3, since the wavefunction is exponentially decaying there. 4) The potential in region 4 is higher than that in region 2 since the wavefunction decays on a shorter length scale there.

2b) Could this potential $V(x)$ have a bound state? If so, mark on your $V(x)$ plot what E might lead to such a state and justify why it is bound. If not, justify why there is no bound state.

Answer: A bound state is one where the wavefunction is localized and decays exponentially at large and small x. For this to occur, we need the particle energy to be less than $V(x)$ as $x \to \pm\infty$. It is possible that such a state can exist, with an example being the energy E_2 shown on the plot. Of course, to truly confirm that one does exiss we would have to use the solution to the time-independent Schrodinger equation in all regions and confirm that there is solution that matches the boundary conditions at all interfaces.

Problem 3: (15 pts)

Consider the "potential cliff"

$$V(x) = V_0 \text{ for } x < 0 \qquad V(x) = -V_0 \text{ for } x \geq 0 \tag{14}$$

a particle of energy $E = \frac{5}{3}V_0$ is incident on the cliff from the left (i.e., coming from $x = -\infty$). There is no particle incident from the right (i.e., coming from $x = \infty$).

3a) Write down the solutions to the time-independent Schrodinger Equation in regions $x < 0$ and $x \geq 0$.

Answer: This problem is very similar to HW#4, problem 3 and problem 4. In both region 1 ($x < 0$) and region 2 ($x > 0$) we have $E > V(x)$ so the solution is oscillating (complex exponential) in each. In region 1 the solution is

$$\psi_1(x) = Ae^{ik_1x} + Be^{-ik_1x} \quad \text{with } k_1 = \left[\frac{2m[E - V_0]}{\hbar^2}\right]^{1/2} \tag{15}$$

and in region 2

$$\psi_1(x) = Ce^{ik_2x} + De^{-ik_2x} \quad \text{with } k_2 = \left[\frac{2m[E - (-V_0)]}{\hbar^2}\right]^{1/2} \tag{16}$$

The problem states that there is no wave incident from the right, so we set $D = 0$. Using the value given, $E_0 = 5/3V_0$ we find

$$k_1 = \left[\frac{4mV_0}{3\hbar^2}\right]^{1/2} \quad \text{and } k_2 = \left[\frac{8mV_0}{3\hbar^2}\right]^{1/2} = 2k_1 \tag{17}$$

We sanity check that $k_2 > k_1$ since the potential is lower in region 2 (i.e., the particle "moves faster" as it "falls down" the cliff).

3b) What is the probability that the particle is transmitted through the cliff? Write your answer as a pure number (i.e., with no variables).

Answer: The transmission probability is given by

$$T = \frac{k_2}{k_1}\frac{|C|^2}{|A|^2} \tag{18}$$

The factor of k_2/k_1 results from the fact that velocity is not the same in regions 1 and 2. This was discussed in class and in the review notes (and on the problem 4 of HW#4).

Applying the boundary conditions at $x = 0$ we have

$$B.C.\#1: \psi_1(x=0) = \psi_2(x=0) \rightarrow A + B = C \tag{19}$$

$$B.C.\#2: \frac{\partial \psi_1}{\partial x}(x=0) = \frac{\partial \psi_2}{\partial x}(x=0) \rightarrow ik_1(A - B) = ik_2C \tag{20}$$

The B.C.#2 implies $A - B = (k_2/k_1)C$ so adding this to the B.C. #1 gives

$$2A = (1 + \frac{k_2}{k_1}C) \rightarrow C = \frac{2A}{(1 + k_2/k_1)} = \frac{2Ak_1}{k_2 + k_1} \tag{21}$$

So the transmission coefficient is

$$T = \frac{k_2}{k_1}\frac{|C|^2}{|A|^2} = \frac{4k_1k_2}{(k_2 + k_1)^2} \tag{22}$$

and since $k_2 = 2k_1$, we have $\boxed{T = 8/9}$. We sanity check that $T \leq 1$ as it must be.

Problem 4: (15 pts)

A particle is in the symmetric infinite square well, with a potential defined as

$$V(x) = 0 \quad \text{for } -a < x < a \qquad V(x) = \infty \quad \text{otherwise} \tag{23}$$

An experimenter puts the particle into an initial state given by the wavefunction

$$\Psi(x, t = 0) = A \quad \text{for } -a/2 < x < a/2 \tag{24}$$

$$\Psi(x, t = 0) = 0 \quad \text{otherwise} \tag{25}$$

where A is a constant. You don't have to rederive the eigenstates of the infinite square well if you know them (though make sure they are consistent with the way $V(x)$ is defined here).

4a) What is the probability that a energy measurement at $t = 0$ returns E_1, the ground state energy (i.e., lowest allowed energy)? Your answer should be a number (i.e., not involve variables a or A).

Answer: This problem is similar to problem 3 on HW#3. Recall that any wavefunction can be written as a superposition of energy eigenstates

$$\psi(x) = \sum_n c_n \psi_n(x) \tag{26}$$

The probability we measure the eigenvalue associated with eigenstate ψ_n is given by $|c_n|^2$, the absolute value squared of the coefficient. We determine c_n using Fourier's trick

$$c_n = \int_{-\infty}^{\infty} \psi(x)\psi_n(x)dx \tag{27}$$

In this problem, the particle is put in the state $\psi(x) = A$ for $-a/2 < x < a/2$. We first determine A by normalization

$$\int_{-a/2}^{a/2} |A|^2 dx = |A|^2 a = 1 \rightarrow |A| = \frac{1}{\sqrt{a}} \tag{28}$$

(where again we set the arbitrary complex phase $\delta = 0$). We also need the eigenstates for the particle in the symmetric infinite well, which come in two forms, even and odd. We determined these in HW#3 problem 2. The ground state is the one with only one bump (and no nodes) so is even (a cosine)

$$\psi_1(x) = \sqrt{\frac{2}{2a}} \cos\left(\frac{\pi x}{2a}\right) \tag{29}$$

We were careful to head the warning in the problem about how the potential was defined here. The length of this box is $L = 2a$, so the normalization constant out front is $\sqrt{2/2a}$. Also we sanity check that we solution we wrote down is one that gives us the correct boundary conditions: $\psi_1(x = -a) = 0$ and $\psi_1(x = a) = 0$.

Now we can determine the coefficient of the ground state, c_1, from Fourier's trick

$$c_1 = \int_{-a/2}^{a/2} \left[\frac{1}{\sqrt{a}}\right] \frac{1}{\sqrt{a}} \cos\left(\frac{\pi x}{2a}\right) dx = \frac{1}{a}\frac{2a}{\pi x} \sin\left(\frac{\pi x}{2a}\right)\Big|_{-a/2}^{a/2} = \frac{2}{\pi}[\sin(\pi/4) - \sin(-\pi/4)] = \frac{4}{\pi}\sin(\pi/4) \tag{30}$$

where we used the fact that $\sin(-\theta) = -\sin(\theta)$. The sine of $\pi/4$ (or 45 degrees) is $1/\sqrt{2}$ so we have

$$c_1 = \frac{4}{\sqrt{2}\pi} \rightarrow \boxed{|c_1|^2 = \frac{8}{\pi^2}} \tag{31}$$

We sanity check that $|c_1|^2 \leq 1$, as it must be for a probability.

4b) What is the probability that a measurement at $t = 0$ of the particle energy returns E_2, the second lowest allowed energy?

Answer: This is similar to the above problem, except we need the next highest energy eigenstate, which will be an odd (sine) one

$$\psi_2(x) = \sqrt{\frac{2}{2a}} \sin\left(\frac{\pi x}{a}\right) \tag{32}$$

Again we sanity check that we wrote down a solution with the right boundary conditions at $x = -a$ and $x = a$. Fourier's trick in this case is

$$c_2 = \int_{-a/2}^{a/2} \left[\frac{1}{\sqrt{a}}\right] \frac{1}{\sqrt{a}} \sin\left(\frac{\pi x}{a}\right) dx \tag{33}$$

We can carry out the integral if we want, but we don't need to. The eigenstate $\psi(2)$ is odd (that is $\psi(x) = -\psi(-x)$ whereas the wavefunction $\psi(x)$ is even. So the integration by symmetry has to be zero

$$|c_2|^2 = 0 \tag{34}$$

4c) If we make an energy measurement at $t = 0$ and get E_1, what is the probability that we will measure E_1 again if we make a second measurement at $t > 0$? Justify your answer.

Answer: If we make an energy measurement and get E_1 we *collapse* the system into the associated eigenstate, so that now $\psi(x) = \psi_1(x)$. The energy eigenstates are the stationary states, so this state does not change in time (other than the factor $e^{-iE_1 t/\hbar}$ which changes the complex phase of wavefunction, but this does not change the absolute magnitude squared of the state). Therefore if we measure energy again, we will get E_1 with 100% probability.

Problem 5: (12 pts)

A free particle (potential $V(x) = 0$ everywhere) has a mass m and a wavefunction at $t = 0$ of

$$\Psi(x, t = 0) = A\left[e^{iax} + ie^{2iax}\right] \tag{35}$$

where A and a are real constants. (Though this wavefunction is technically not normalizable, we can use it to approximate a very long wavetrain that is normalizable.)

5a) Does this particle have a well-defined (i.e., certain) value of momentum? Argue why or why not.

Answer: The states of well-defined (certain) momentum are the momentum eigenstates, which are monochromatic waves

$$\psi_p = Ae^{ikx} \tag{36}$$

Which have eigenvalues $p = \hbar k$. We see that this wavefunction is a *superposition* of two momentum eigenstates with two different momentum eigenvalues, $p_1 \hbar a$ and $p_2 \hbar 2a$. Therefore this state does not have a single, well-defined value of momentum.

Indeed you may recall that in HW#1 problem 2 you worked out the sum of two complex exponentials with different oscillation frequencies. You got a beat pattern, which is not a perfect monochromatic wave. So $\Psi(x, 0)$ is not a momentum eigenstate.

If we still had doubts, we could apply the momentum operator and directly check if the state given is a momentum eigenstate

$$\hat{p}\Psi(x, 0) = -i\hbar A\frac{\partial}{\partial x}\left[e^{iax} + ie^{2iax}\right] = -i\hbar\left[iae^{iax} - 2ae^{2iax}\right] \tag{37}$$

We see that $\hat{p}\Psi(x, 0) \neq p_0\Psi(x, 0)$ for any constant p_0, therefore this is *not* a momentum eigenstate. The eigenstates are the only ones with well-defined (certain) values for the observable (see HW#4 problem 2).

5b) Does this particle have a well-defined (i.e., certain) value of energy? Argue why or why not.

Answer: The energy eigenstates for the free particle are also the monochromatic waves

$$\psi_E = Ae^{ikx} \tag{38}$$

with energy eigenvalues of $E = \hbar^2 k^2/2m$. Therefore the given wavefunction is the superposition of two eigenstates with two different energies

$$E_1 = \frac{\hbar^2 a^2}{2m} \quad \text{and } E_2 = \frac{4\hbar^2 a^2}{2m} \tag{39}$$

and does not have a single, well-defined energy. If we wanted, we could proceed as above and apply the Hamiltonian operator to show that indeed $\Psi(x, 0)$ is not an eigenstate of the Hamiltonian.

5c) Write down the wavefunction at a later time $\Psi(x, t)$.

Answer: We know that any energy eigenstate evolves in time just in the complex phase

$$\Psi_E(x, t) = \Psi_E(x)e^{-iEt/\hbar} \tag{40}$$

The given wavefunction is a superposition of two energy eigenstates with different energies, each of which evolve according to a different phase, so the time evolution of this state is

$$\boxed{\Psi(x, t) = A\left[e^{iax}e^{-iE_1 t/\hbar} + ie^{2iax}e^{-iE_2 t/\hbar}\right]} \tag{41}$$

where E_1 and E_2 are given above. Review HW#2 problem 2 where we did a similar time evolution for the superposition of two energy eigenstates for the particle in an infinite square well.

5d) What is the probability that at some time $t > 0$ we measure the particle to be near $x = 0$ (i.e., between $x = 0$ and $x = 0 + dx$)?

Answer: The quantity $|\Psi(x,t)|^2$ is defined as the probability of finding a particle between x and $x+dx$. So the problem is asking us to determine $|\Psi(x,t)|^2$ at $x=0$. From the above solution for $\Psi(x,t)$ we have

$$\Psi(x=0,t) = A\left[e^{-iE_1 t/\hbar} + ie^{-iE_2 t/\hbar}\right] \tag{42}$$

And so multiplying this by its complex conjugate

$$\begin{aligned}
|\Psi(x=0,t)|^2 &= A\left[e^{-iE_1 t/\hbar} + ie^{-iE_2 t/\hbar}\right] A^*\left[e^{iE_1 t/\hbar} - ie^{iE_2 t/\hbar}\right] &(43)\\
&= |A|^2\left[1 - ie^{-iE_1 t/\hbar}e^{iE_2 t/\hbar} + ie^{-iE_2 t/\hbar}e^{iE_1 t/\hbar} + 1\right] &(44)\\
&= |A|^2\left[2 + ie^{i(E_1-E_2)t/\hbar} - ie^{i(E_1-E_2)t/\hbar}\right] &(45)\\
&= 2|A|^2\left[1 + \sin\left(\frac{(E_1-E_2)t}{\hbar}\right)\right] &(46)
\end{aligned}$$

where in the last step we used Euler's formula to write the difference of the complex exponentials as a sine, which makes it explicit that this function is real (as it must be). So the probability of finding the particle near $x=0$ oscillates over time between zero and a value of $4|A|^2$.