EECS 16A Designing Information Devices and Systems I
Fall 2017 Midterm 1

Exam Location: 155 Dwinelle

PRINT your student ID: ____

PRINT AND SIGN your name: ____________________, ____________________ ______________________
(last name) (first name) (signature)

PRINT your discussion section and GSI(s) (the one you attend): ____

Name and SID of the person to your left: ____

Name and SID of the person to your right: ____

Name and SID of the person in front of you: ____

Name and SID of the person behind you: ____

1. **What did you do over the summer? (1 Point)**

2. **What activity do you really enjoy? (1 Point)**

Do not turn this page until the proctor tells you to do so. You may work on the questions above.

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Extra page for scratchwork.
If you want any work on this page to be graded, please refer to this page on the problem's main page.

3. Fun Times with Inverses (7 Points)

Consider the following matrix $\mathbf{A}$.

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}$$

(a) (5 Points) *Calculate its inverse* $\mathbf{A}^{-1}$ if it exists.

(b) (2 Points) If an $n \times n$ matrix $\mathbf{A}$ does not have an inverse, is the dimension of $\mathbf{A}$'s column space (i.e., the span of the column vectors of $\mathbf{A}$) less than, greater than, or equal to n? *Circle your answer(s).*

LESS THAN **GREATER THAN** **EQUAL TO**

4. Mechanical Basis (4 Points)

Given $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 1 \\ x \end{bmatrix}$, *find a value of* x such that these three vectors do **not** form a basis for $\mathbb{R}^3$. Show why your choice of x makes it so these vectors do **not** form a basis for $\mathbb{R}^3$.

5. Mechanical Eigenvalues (5 Points)

Consider the matrix $\mathbf{A}$ below.

$$\mathbf{A} = \begin{bmatrix} 1 & a \\ b & 1 \end{bmatrix}$$

Find the eigenvalues of $\mathbf{A}$ in terms of a and b.

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6. Operation Crypto (17 Points)

Agent 16A, you have been tasked with an important mission. The Berkeley Intelligence Agency has been collecting important information on the other engineering schools (known as the adversary) using our field agents. Through their last reports, we have found out that our adversaries have been secretly sharing milk tea recipes through cryptographic ciphers in the form of Matrix Transformations, and we want to find out what these recipes are!

The adversary uses $n \times n$ preshared key matrices to encode plaintext vectors $\vec{p} = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix}$.

$p_1,\ p_2,\ \ldots, p_n$ are real numbers from 0 to 25 that map to A through Z. That is, 0 maps to A, and 25 maps to Z.

To get the ciphertext vector, they multiply their key matrix

$$\mathbf{K} = \begin{bmatrix} k_{11} & \cdots & k_{1n} \\ \vdots & \ddots & \vdots \\ k_{n1} & \cdots & k_{nn} \end{bmatrix},$$

where all the elements in $\mathbf{K}$ are real numbers, with their plaintext vector $\vec{p}$ to get

$$\mathbf{K}\vec{p} = \vec{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix},$$

where the elements in $\vec{c}$ are real numbers that don't necessarily map to anything. To a passive observer, the entries in $\vec{c}$ look like nonsense.

(a) (4 Points) In a hypothetical scenario, MIT wants to send Stanford a milk tea recipe $\vec{p}$ by sending $\vec{c} = \mathbf{K}\vec{p}$. *What has to be true about* $\mathbf{K}$ *for Stanford be able to recover* $\vec{p}$*? In that case, how would Stanford recover the plaintext recipe* $\vec{p}$ *(decrypt the ciphertext vector)?*

(b) (4 Points) Agent 16A, your first objective is to find the key. Your mission is to perform what is called a *Known Plaintext Attack*, where the attacker (in this case, us) knows a plaintext vector $\vec{p}$ and its corresponding ciphertext vector $\vec{c}$, which we will call a plaintext/ciphertext pair. One field agent was able to uncover a plaintext/ciphertext pair:

$$\left(\vec{p}_1 = \begin{bmatrix} 1 \\ 14 \end{bmatrix}, \vec{c}_1 = \begin{bmatrix} 29 \\ 72 \end{bmatrix}\right)$$

For a 2×2 key matrix $\mathbf{K}$, *is it possible to fully determine* $\mathbf{K}$ *using just this pair? Briefly justify your answer.*

Hint: How many elements are there in $\mathbf{K}$?

(c) (4 Points) In general, for an $n \times n$ key matrix $\mathbf{K}$, *at least how many plaintext/ciphertext vector pairs do we need to be able to fully determine* $\mathbf{K}$*?*

(d) (5 Points) Through her dedication to the mission, one of the field agents was able to uncover a second plaintext/ciphertext pair. *Find the key matrix* $\mathbf{K}$*, such that* $\vec{c}_i = \mathbf{K}\vec{p}_i$. Don't let her efforts be in vain! For your convenience, here are both plaintext/ciphertext pairs.

$$\left(\vec{p}_1 = \begin{bmatrix} 1 \\ 14 \end{bmatrix}, \vec{c}_1 = \begin{bmatrix} 29 \\ 72 \end{bmatrix}\right), \left(\vec{p}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{c}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$$

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7. 3D Imaging (18 Points)

In lab, you built a 2D imaging system. In this problem, we will explore how to build a 3D tomography system. We will be trying to image voxels, which are a generalization of pixels in 3 dimensions. Each voxel has a value associated with it that represents how much light passes through it. This is equivalent to how dark or light the voxel appears.

We would like to image a $2 \times 2 \times 2$ structure. Each incident light ray passes all the way through the structure. Figure 7.1 demonstrates a few different example rays passing through our 3D structure. Note that lighter colored voxels represent voxels through which the light ray passes.

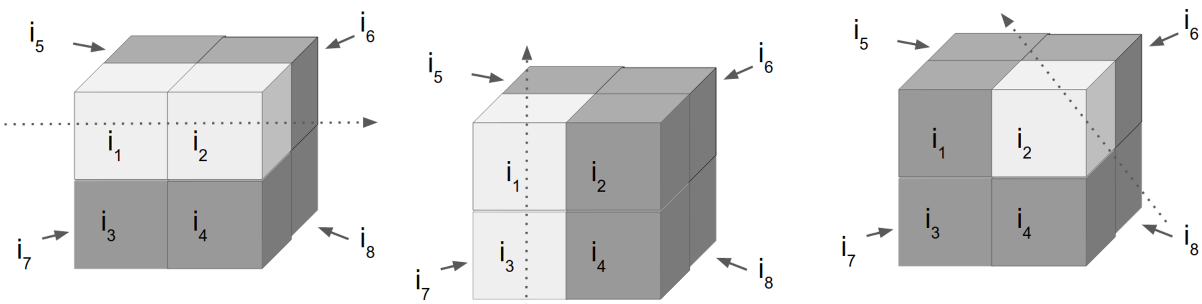

Figure 7.1: Three different incident rays.

As in the imaging lab, we will represent the three-dimensional $2 \times 2 \times 2$ structure as one column vector $\vec{i}$ with 8 elements. We will then apply rays that pass through the object represented by row vectors $\vec{h}_k^T$. Let the matrix $\mathbf{H}$ be the matrix whose row vectors are $\vec{h}_k^T$. Finally, we will measure the vector $\vec{s}$ where $\vec{s} = \mathbf{H}\vec{i}$.

(a) (5 Points) Suppose that we have the matrix $\mathbf{H}$ shown below. *Find the null space of this matrix* $\mathbf{H}$.

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(b) (5 Points) Find two objects that would give the same $\vec{s}$ when measured using the mask matrix $\mathbf{H}$ from part (a). *Write down the corresponding $\vec{i}$ vectors for the two objects.* Note that the two $\vec{i}$ vectors can contain any real number you would like them to, but the two vectors cannot be equal to each other. (For example $\vec{i}_1 = \vec{i}_2 = \vec{0}$ is not a valid solution.)

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(c) (8 Points) Our matrix $\mathbf{H}$ only has four rows corresponding to four measurements. *Add another four rows to the original* $\mathbf{H}$ *matrix to make a new mask matrix* $\mathbf{H}'$ *of dimensions* 8×8 and select the entries of the additional four rows such that we can always uniquely reconstruct the imaged object. Each row of your new matrix can have at most two 1's in it, and the rest of the entries must be 0.

8. The Art of Proving (14 Points)

Dr. Alon and Dr. Sahai are two passionate scientists who are currently working on linear algebra. They have been working for a long time and have hit some major roadblocks. Help them out by proving some things that they are stuck on!

Let $\mathbf{A}$ and $\mathbf{B}$ be two matrices with dimensions, such that $\mathbf{AB}$ is valid.

(a) (4 Points) Show that *if $\vec{v}$ is in the null space of* $\mathbf{B}$*, then $\vec{v}$ is in the null space of* $\mathbf{AB}$.

(b) (4 Points) Show that *if λ is an eigenvalue of an $n \times n$ matrix* $\mathbf{A}$*, then for all $c \in \mathbb{R}$, $\lambda' = \lambda + c$ is an eigenvalue of the matrix* $\mathbf{A} + c\mathbf{I}$, where $\mathbf{I}$ is the $n \times n$ identity matrix.

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(c) (6 Points) Suppose that $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ matrices that share the same n distinct eigenspaces. That is, there exist n non-zero vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$, such that $\mathbf{A}\vec{v}_i = \lambda_{A,i}\vec{v}_i$ and $\mathbf{B}\vec{v}_i = \lambda_{B,i}\vec{v}_i$, where $\lambda_{A,i}$ is not necessarily equal to $\lambda_{B,i}$. Then, prove that *the matrices* $\mathbf{A}$ *and* $\mathbf{B}$ *commute*, that is, $\mathbf{AB} = \mathbf{BA}$.

Hint: Choose an appropriate basis.

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9. Rabbits, Foxes, and the Circle of Life (21 Points)

If rabbits are such notoriously fast breeders, why haven't we all been crushed under a (warm, comfortable) mountain of rabbits by now? Well, consider the hungry foxes...

Let's examine the case of Tilden Park, circa 1000 CE. This vast beautiful space is initially filled with 200 foxes and 1000 rabbits. Since rabbits like to feast on the pleantiful greenery, the population of rabbits grows by 10% each month. Every month, 40% of the foxes either die or leave the park. The population of foxes increases by 20% of the population of rabbits each month. Similarly the population of rabbits decreases by 20% of the fox population each month. This can be summarized by the system shown below, where $f[t]$ and $r[t]$ represent the number of foxes and rabbits in the park each month t.

$$\begin{bmatrix} f[t+1] \\ r[t+1] \end{bmatrix} = \begin{bmatrix} 0.6 & 0.2 \\ -0.2 & 1.1 \end{bmatrix} \begin{bmatrix} f[t] \\ r[t] \end{bmatrix}$$

In this problem, we will use linear algebra to explore the predator-prey relationship to figure out if we should be submerged in rabbits, on the run from armies of foxes, or in some peaceful equilibrium state.

(a) (8 Points) We want to know what will happen to the populations of the two species as time goes on. Will the population numbers converge?

Note: You do not need to find what the populations converge to, just whether they converge or not. You may or may not find it useful to know that $1.7^2 = 2.89$.

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(b) (5 Points) Assuming that there is some known total populaton of foxes and rabbits in the year 2000, *calculate what fraction of that total population is rabbits.* You can assume that 1000 years is a good approximation for an infinite amount of time.

(c) (8 Points) In the far future, a curious child digs up a strange fossil in the park. It seems like Ancient Dino-foxes once inhabited the park! Using advanced future Zoologic-Mathematics, graduate students from the University of MegaCalifornia, Berkeley derive the following eigenvalue/eigenvector pairs describing the species interactions from the fossils:

$$\left(\lambda_1 = 1, \vec{v}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}\right), \left(\lambda_2 = 0.5, \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$$

Reconstruct the state transition matrix.

Doodle page!
Draw us something if you want or give us suggestions, compliments, or complaints.
You can also use this page to report anything suspicious that you might have noticed.

EECS 16A Designing Information Devices and Systems I
Fall 2017 Midterm 1 Instructions

Read the following instructions before the exam.

You have 120 minutes for the exam. There are 9 problems of varying numbers of points. The problems are of varying difficulty, so pace yourself accordingly and avoid spending too much time on any one question until you have gotten all of the other points you can.

There are 88 points possible on this exam. Partial credit will be given for substantial progress on each problem.

Distribution of the points:

Problem	1	2	3	4	5	6	7	8	9
Points	1	1	7	4	5	17	18	14	21

The exam is printed double-sided. Do not forget the problems on the back sides of the pages!

There are 24 pages on the exam, so there should be 12 sheets of paper in the exam. Notify a proctor immediately if a page is missing. **Do not tear out or remove any of the pages. Do not remove the exam from the exam room.**

Write your name and your student ID on each page before time is called. If a page is found without a name and a student ID, we are not responsible for identifying the student who wrote that page.

You may consult ONE handwritten 8.5” $\times$ 11” note sheet (front and back). No phones, calculators, tablets, computers, other electronic devices, or scratch paper are allowed. **No collaboration is allowed, and do not attempt to cheat in any way. Cheating will not be tolerated.**

Please write your answers legibly in the spaces provided on the exam; we will not grade outside a problem’s designated space unless you specifically tell us where to find your work. In general, show all of your work in order to receive full credit.

If you need to use the restrooms during the exam, bring your student ID card, your phone, and your exam to a proctor. You can collect them once you return from the restrooms.

Our advice to you: if you can’t solve the problem, state and solve a simpler one that captures at least some of its essence. You might get some partial credit, and more importantly, you will perhaps find yourself on a path to the solution.

Good luck!