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Math54 Midterm I, Fall 2016

This is a closed book exam. Everyone is allowed a one-page cheat-sheet but no calculators. You need to justify every one of your answers unless you are asked not to do so. Completely correct answers given without justification will receive little credit. Problems are not necessarily ordered according to difficulties.

Problem	Maximum Score	Your Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	

Your Name: [redacted]

Your GSI:

Your SID:

Name and SID: [redacted]

1. Solve linear systems of equations $A\,x = b$, where

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

using the row reduction algorithm.

$$[A \vdots b] = \begin{matrix} (1) \\ (2) \\ (3) \end{matrix} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 0 \end{bmatrix} \underset{(3)=(3)-(1)}{\overset{(2)=(2)-(1)}{\sim}} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 2 & 0 \end{bmatrix}$$

$$\text{swap } (2) \,\&\, (3) \sim \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \underset{(1)=(1)-(3)}{\overset{(2)=(2)-2(3)}{\sim}} \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$(1)=(1)-(2) \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

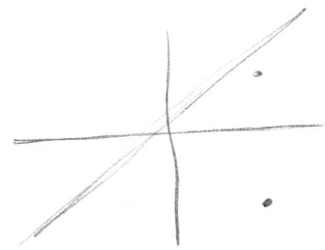
$$\boxed{\begin{aligned} x_1 &= 1 \\ x_2 &= -2 \\ x_3 &= 1 \end{aligned}} \qquad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \boxed{\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}}$$

↑ answer

check: $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ ✓

2. Let $\mathbf{T} : \mathcal{R}^3 \to \mathcal{R}^3$ be the linear transformation that reflects each vector through the plane $x_2 = 0$. That is

$$\mathbf{T}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ -x_2 \\ x_3 \end{pmatrix}.$$



Find the standard matrix of $\mathbf{T}$. $Tx = b$

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ -x_2 \\ x_3 \end{pmatrix}$$

3x3 3x1 3x1

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_2 \\ x_3 \end{bmatrix}$$

$$\boxed{T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

I got this by inspection. Only x_2 gets its sign flipped, so I just multiplied the 2^{nd} row of the elementary matrix by -1.

3. Mark each statement **True** or **False**. Do not need to justify your answers.

(a) In order for a matrix B to be the inverse of A, both equations

$$A\,B = I \quad \text{and} \quad B\,A = I$$

must be true.

(b) Each elementary matrix is invertible.

(c) Let A and $B \in \mathcal{R}^{n \times n}$ be both invertible. Then their product $A\,B$ is also invertible with inverse $A^{-1}\,B^{-1}$.

(d) If $A \in \mathcal{R}^{n \times n}$ is invertible. Then the equation $A\mathbf{x} = \mathbf{b}$ is consistent for *each* $\mathbf{b} \in \mathcal{R}^n$.

(a) True — Can only get identity matrix if both A and B are square matrices

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3 × 4 4 × 3 3 × 3

(b) True

(c) False

(d) True

Name and SID: [redacted]

4. Find a basis for the column space of A, where

$$A = \begin{pmatrix} 0 & 2 & 6 \\ 2 & 2 & 16 \\ -1 & 0 & -5 \end{pmatrix}.$$

$A = \begin{pmatrix} 0 & 2 & 6 \\ 2 & 2 & 16 \\ -1 & 0 & -5 \end{pmatrix}$ Swap (1 & 2) $\sim \begin{pmatrix} 2 & 2 & 16 \\ 0 & 2 & 6 \\ -1 & 0 & -5 \end{pmatrix}$ (1) = (1)/2, (2) = (2)/2, (3) = (3) · −1 $\sim \begin{bmatrix} 1 & 1 & 8 \\ 0 & 1 & 3 \\ 1 & 0 & 5 \end{bmatrix}$

(3) = (3) − (1) $\sim \begin{bmatrix} 1 & 1 & 8 \\ 0 & 1 & 3 \\ 0 & -1 & -3 \end{bmatrix}$ (3) = (3) + (2) $\sim \begin{bmatrix} 1 & 1 & 8 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ ■ = pivot

basis = columns that have a pivot, which are columns 1 and 2.

So the basis for the column space for the original matrix A (which = the row reduced form)

$$= \left\{ \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} \right\}$$

5. (a) Use Cramer's Rule to solve

$$A\mathbf{x} = \mathbf{b}, \quad \text{where} \quad A = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

(b) Compute the determinant of

$$A = \begin{pmatrix} 0 & 0 & 6 \\ 0 & 2 & 16 \\ -1 & 0 & -5 \end{pmatrix}.$$

$a_1 \quad a_2$

$A = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$

$\det(A) = (3)(1) - (2)(1) = 1$

$\det(b\ a_2) = \begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix} = (1)(3) - (-1)(2) = 5$

$\det(a_1\ b) = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = (1)(-1) - (1)(1) = -2$

$x_1 = \dfrac{\det(b\ a_2)}{\det A} = \dfrac{5}{1} = 5$

$x_2 = \dfrac{\det(a_1\ b)}{\det A} = -2 = -2$

$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$ ← answer

check: $\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 5 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ ✓

(b)

$\det A = \begin{vmatrix} 0 & 0 & 6 \\ 0 & 2 & 16 \\ -1 & 0 & -5 \end{vmatrix}$

swap (1) and (3)

$\det A = -\begin{vmatrix} -1 & 0 & -5 \\ 0 & 2 & 16 \\ 0 & 0 & 6 \end{vmatrix}$ Triangular

$\det A = -(\text{product of diagonal})$

$= -(-1 \cdot 2 \cdot 6)$

$= -(-12) = 12$