Midterm 1 Solutions

Physics 7B
Lectures 2,3

Fall 2015

Problem 1

Let R_h be the radius of the hole.

$$\Delta R_h = \frac{2}{3}R\alpha\Delta T \Rightarrow R_h = \frac{2}{3}R + \Delta R_h = \frac{2}{3}R(1+\alpha\Delta T)$$

(4 points)
In order for the marble to fit through the hole, $R_h \geq R$. (2 points)
Therefore, we want

$$\frac{2}{3}R(1+\alpha\Delta T) \geq R \Rightarrow 1+\alpha\Delta T \geq \frac{3}{2} \Rightarrow \Delta T \geq \frac{1}{2\alpha}.$$

Therefore, in order for the marble to fit through the hole, oone must increase the steel temperature by at least $\frac{1}{2\alpha}$. (4 points)

Problem 2

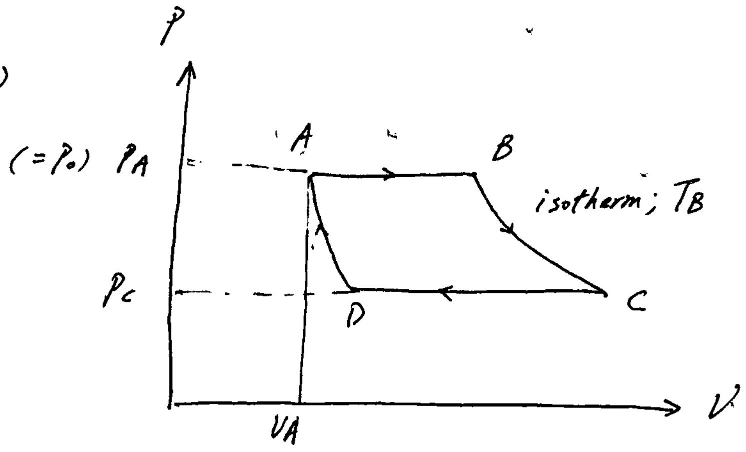
Let M be the mass of ice that melts. Then $Q = ML$. (2 points)
The heat transfered is equal to the kinetic energy of the bullet plus the heat conducted due to the difference in temperature (2 points).
The kinetic energy of the bullet is equal to $\frac{mv^2}{2}$ (2 points)
The heat transfered is due to the temperature gradient is equal to $mc_B(T_B - 0°C)$. (2 points)
Thus

$$\frac{mv^2}{2} + mc_B(T_B - 0°C) = ML \Rightarrow M = \frac{mv^2}{2L} + \frac{mc_B}{L}(T_B - 0°C)$$

(2 points)

3)

a)



- adiabat (D→A) steeper than isotherm (B→C)
- ½ points taken off for each wrong process
- ½ points taken off for each incomplete description (not labelling ABCD etc)

b) Ideal diatomic gas ⇒ $d=5$; $\gamma = \frac{7}{5}$

I) $B \to C$ is an isotherm

$$\Rightarrow T_c = T_B$$

II) To calculate T_D, first find T_A.

By ideal gas law,

$$T_A = \frac{P_A V_A}{nR} = \frac{P_A V_A}{(1)R} \qquad (n=1)$$

A−D is an adiabatic process; hence

$$P_A V_A^{\gamma} = P_D V_D^{\gamma}$$

$$\Rightarrow P_A^{1-\gamma} T_A^{\gamma} = P_D^{1-\gamma} T_D^{\gamma} \qquad \left(V \sim \frac{T}{P}\right)$$

$$T_D = T_A \left(\frac{P_A}{P_D}\right)^{\frac{1-\gamma}{\gamma}}$$

$$= \frac{P_A V_A}{R} \left(\frac{P_A}{P_C}\right)^{-\frac{2}{7}} \qquad (P_D = P_c)$$

$$= \frac{P_A^{+\frac{5}{7}} P_c^{\frac{2}{7}} V_A}{R}$$

- 1 point for T_c
- 4 points for T_D
  - +1 for $PV^{\gamma} = $ const correctly applied to A−D process
  - +1 for using ideal gas law correctly
  - +1 for γ / d
  - +1 for final answer

c) Grading policy:
- 1 point for each of the 12 processes
  - $+\frac{1}{2}$ for correct process
  - $+\frac{1}{2}$ for correct answer
- $\frac{1}{2}$ points taken off for each wrong / missing final answer
- ($\Delta E, Q, W$ for entire cycle)
- full points given if quantities for the whole cycle are given without steps for each process, provided correct reasoning & answer

	ΔE	Q	W
$A \to B$	$\frac{d}{2}nR\Delta T = \frac{5}{2}R(T_B - \frac{P_AV_A}{R})$	$nC_p\Delta T = \frac{7}{2}R(T_B - \frac{P_AV_A}{R})$	$Q - \Delta E = R(T_B - \frac{P_AV_A}{R})$
$B \to C$	$\frac{d}{2}nR\Delta T = 0$	$W = RT_B \ln(\frac{P_A}{P_c})$	$nRT\ln(\frac{V_c}{V_B}) = RT_B\ln(\frac{P_A}{P_c})$
$C \to D$	$\frac{d}{2}nR\Delta T = \frac{5}{2}R(\frac{P_A^{\frac{5}{7}}P_c^{\frac{2}{7}}V_A}{R} - T_B)$	$nC_p\Delta T = \frac{7}{2}R(\frac{P_A^{\frac{5}{7}}P_c^{\frac{2}{7}}V_A}{R} - T_B)$	$Q - \Delta E = R(\frac{P_A^{\frac{5}{7}}P_c^{\frac{2}{7}}V_A}{R} - T_B)$
$D \to A$	$\frac{d}{2}nR\Delta T = \frac{5}{2}V_A(P_A - P_A^{\frac{5}{7}}P_c^{\frac{2}{7}})$	0	$-\Delta E = -\frac{5}{2}V_A(P_A - P_A^{\frac{5}{7}}P_c^{\frac{2}{7}})$

For the whole process(cycle),

- $\Delta E = 0$ (full points if state this directly IF reasoning is given; if not just 1 point)
- $Q = RT_B \ln\frac{P_A}{P_c} + \frac{7}{2}V_A(P_A^{\frac{5}{7}}P_c^{\frac{2}{7}} - P_A)$
- $W = RT_B \ln\frac{P_A}{P_c} + \frac{7}{2}V_A(P_A^{\frac{5}{7}}P_c^{\frac{2}{7}} - P_A) = Q$

NB

$$V_B = \frac{RT_B}{P_A}$$

$$V_c = \frac{RT_c}{P_c}$$

Problem 4

Solution

During a particular interval, heat in the amount of Q_0 must be extracted from the house to maintain its temperature.

During the same interval, Q_1 must be extracted to accomplish the required cooling. We can regard Hot water tank Ⓔ and surrounding as a heat engine, and if it is an ideal heat engine

$$W_{net} = \frac{T_W - T_S}{T_w} \mathrm{Q}_1 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{ 10 pts}$$

We can regard House © and surrounding as a refrigerator and the work from the engine is used to cool the house.

$$W_{net} = \frac{T_S - T_H}{T_H} \mathrm{Q}_0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{ 10 pts}$$

We can find that the minimum Q_1 is determined by

$$\mathrm{Q}_1 = \frac{T_W}{T_H} \frac{T_S - T_H}{T_w - T_S} \mathrm{Q}_0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{ 20 pts}$$

(a) States 1 and 2 are at the same temperature T, so they lie on a common isotherm. The simplest thermal process from 1 to 2 would be an **isothermal process**. The diagram for this process is shown below.

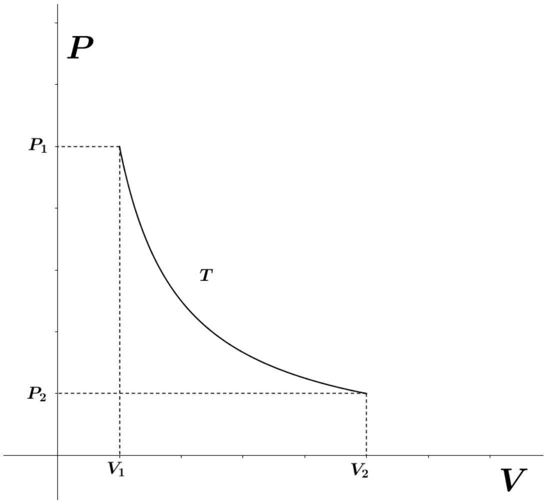


Figure 1: PV Diagram for Isothermal Process

Assume this process is done reversibly. Then the entropy change is

$$\Delta S = \int_{\mathcal{P}} \frac{dQ}{T} \tag{1}$$

where $\mathcal{P}$ is the isothermal process. For an isothermal process, the change in temperature is $\Delta T = 0$, which implies that the change in internal energy of the gas is $\Delta E = 0$. Thus, the first law tells us

$$\Delta E = Q - W = 0 \Rightarrow Q = W.$$

In differential form, this becomes $dQ = dW = PdV$. So the change in entropy is

$$\begin{aligned}\Delta S &= \int_{\mathcal{P}} \frac{PdV}{T} \\ &= \int_{\mathcal{P}} \frac{nR}{V} dV \\ &= nR \int_{V_1}^{V_2} \frac{dV}{V} \\ &= \boxed{R \ln \left(\frac{V_2}{V_1} \right)},\end{aligned}$$

where we have used the ideal gas law to write $\dfrac{P}{T} = \dfrac{nR}{V}$ and noted that $n = 1$ as stated in the problem.

(b) The new PV diagram is shown in the diagram below. Note that the pressure and volume for point $0 = (P_0, V_0)$ satisfies $P_0 = P_2$ and $V_0 = V_1$.

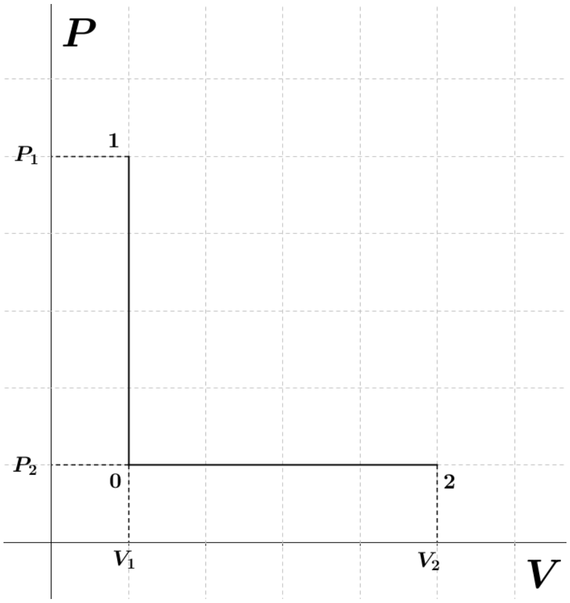


Figure 2: PV Diagram for Two-Step Process

We wish to calculate the entropy changes $\Delta S_{1\to 0}$ and $\Delta S_{0\to 2}$. The first, an isovolumetric process, satisfies $W = \int p dV = 0$, because there is no change in volume. Thus, the first law gives us

$$\Delta E = Q - W = Q \Rightarrow dE = dQ$$

But note that the energy of an ideal gas satisfies

$$E = \frac{d}{2} nRT \Rightarrow dE = \frac{d}{2} nRdT$$

where d is the number of degrees of freedom for the gas. Thus, the change in entropy is

$$\begin{aligned} \Delta S_{1\to 0} &= \int\limits_{1\to 0} \frac{dQ}{T} \\ &= \int\limits_{1\to 0} \frac{dE}{T} \\ &= \frac{d}{2} nR \int_{T_1}^{T_0} \frac{dT}{T} \\ &= \frac{d}{2} R \ln\left(\frac{T_0}{T_1}\right) \\ &= \boxed{\frac{d}{2} R \ln\left(\frac{T_0}{T}\right),} \end{aligned}$$

once again noting that $n = 1$.

The second entropy change $\Delta S_{0\to 2}$ occurs during an isobaric process. In this case, we have according to the first law

$$dQ = dE + dW = \frac{d}{2} nRdT + PdV$$

So the entropy change is

$$\begin{aligned}
\Delta S_{0\to 2} &= \int\limits_{0\to 2} \frac{\frac{d}{2} nRdT + PdV}{T} \\
&= \frac{d}{2} nR \int\limits_{0\to 2} \frac{dT}{T} + \int\limits_{0\to 2} \frac{PdV}{T} \\
&= \frac{d}{2} nR \int_{T_0}^{T_2} \frac{dT}{T} + nR \int_{V_0}^{V_2} \frac{dV}{V} \\
&= \frac{d}{2} nR \ln\left(\frac{T_2}{T_0}\right) + nR \ln\left(\frac{V_2}{V_0}\right),
\end{aligned}$$

where we have once again used the ideal gas law to write $\frac{P}{T} = \frac{nR}{V}$. Note that $V_0 = V_1$ (the first process was isovolumetric) and $T_2 = T$, so this becomes

$$\Delta S_{0\to 2} = \boxed{\frac{d}{2} R \ln\left(\frac{T}{T_0}\right) + R \ln\left(\frac{V_2}{V_1}\right).}$$

Note that $\Delta S_{1\to 0} + \Delta S_{0\to 2} = \Delta S_{\text{isothermal}}$, which is what we expect.

(c) We have

$$\begin{aligned}
\Delta S_{1\to 3} &= \int\limits_{1\to 3} \frac{dQ}{T} \\
&= \boxed{0,}
\end{aligned}$$

because for an adiabatic process $dQ = 0$.

(d) While free expansion is an adiabatic process, it is not reversible, so we cannot apply the normal formula $\Delta S = \int \frac{dQ}{T}$ and claim $\Delta S = 0$ because $dQ = 0$. Instead, we must find some reversible process that carries us from state 1 to 2. Because entropy is a state variable, only the initial and final states are relevant to the change in entropy.

In part **(a)**, we gave a reversible, isothermal process that carried us from state 1 to 2. We calculated ΔS for that process, hence

$$\Delta S_{\text{free}} = \Delta S_{\text{isothermal}} = \boxed{R \ln\left(\frac{V_2}{V_1}\right).}$$

(a) **5 points total;**

- **1 point** for the identification of the correct thermal process;
- **1 point** for a correct PV diagram;
- **1 point** for setting up entropy integral;
- **1 point** for noting that $dQ = dW$;
- **1 point** for correct answer.

(b) **5 points total;**

- **1 point** for a correct PV diagram;
- **1 point** for setting up first integral correctly (noting that $dE = dQ$);
- **1 point** for correct answer for $\Delta S_{1\to 0}$;
- **1 point** for setting up second integral correctly (noting that $dQ = dE + PdV$);
- **1 point** for correct answer for $\Delta S_{0\to 1}$.

(c) **5 points total;**

- **1 point** for some statement implying that $dQ = 0$ or no heat exchange;
- **1 point** for setting up the entropy integral;
- **3 points** for observing that $dQ = 0$ implies $\Delta S = 0$;

(d) **5 points total;**

- **1 point** for some acknowledgment that the entropy integral fails or some statement that the process is irreversible;
- **2 points** for some set-up of an alternate, reversible pathway OR some set-up of $S = k_B \log \Omega$;
- **2 points** for the correct answer.

Problem 6 (Solution)

Two moles of an ideal gas with molar specific heat $c_v = 5/2R$ *are initially at temperature* T_0 *and pressure* P_0*. Vibrational degrees of freedom can be neglected in this temperature range.*

1. *(3pts) How many degrees of freedom does the gas have? Explain your answer.*

   Recall the average internal energy for n moles of gas is given by:

   $$E = \frac{d}{2} nRT$$

   where d is the number of degrees of freedom for the gas under consideration. The molar heat capacity at constant volume is defined as:

   $$c_v = \frac{1}{n}\left(\frac{\partial Q}{\partial T}\right)_v$$

   Now at constant volume, the first law of thermodynamics, $dE = dQ - dW$ reduces to $dE = dQ$ as no work can be done when $dV = 0$. Thus we may write:

   $$c_v = \frac{1}{n}\left(\frac{\partial E}{\partial T}\right)_v$$

   Plugging in $E = \frac{d}{2}nRT$ we see:

   $$c_v = \frac{d}{2}R$$

   Comparing this result to the c_v given in the problem, we immediately see $d = 5$ and conclude this gas has 5 degrees of freedom.

2. *(2pts) Could this be a monatomic gas? Explain you answer.*

   No, a monatomic gas can have at most 3 quadratic degrees of freedom, namely v_x, v_y, v_z. While it is possible to limit these degrees (ex. confine the gas to a two dimensional plane), it is impossible to add degrees of freedom.

3. *(5pts) Determine the change in internal energy, the temperature change and the work done by the gas when heat Q is added to the gas at constant pressure*

   There is no doubt many ways to arrive at the correct solution, I present only one here. Begin by computing the change in temperature as follows:

   $$c_p = \frac{1}{n}\left(\frac{\partial Q}{\partial T}\right)_p \Rightarrow \Delta T = \frac{Q}{nc_p}$$

now recall for an ideal gas, $c_p = c_v + R$. Thus in this particular case (with $c_v = 5/2R \Rightarrow c_p = 7/2R$) we have:

$$\Delta T = \frac{2Q}{7nR} \Rightarrow \frac{Q}{7R}$$

The change in internal energy is then:

$$\Delta E = \frac{5}{2}nR\Delta T \Rightarrow \frac{5}{7}Q$$

Finally, from the first law, we can compute the work.

$$W = Q - \Delta E \Rightarrow \frac{2}{7}Q$$

Problem 6 (Rubric)

1. (a) 1 pt for writing down a correct definition of molar specific heat
   (b) 1 pt for applying 1st law and recognizing $dW = 0$
   (c) 1 pt for correct result
2. (a) 1 pt correct answer
   (b) 1 pt for identifying it is because monatomic gas cannot have more than 3 degrees of freedom
3. (a) 3 pts for giving valid expressions for $\Delta T, \Delta E, and W$ when Q is added at constant pressure, even if expressions involve variable not explicitly given in the problem.
   (b) 2 pts will additionally be awarded if the expressions are in terms of given variables.