EECS C128/ ME C134
Final
Fri. Dec. 18, 2015
1910-2200 pm

Name:___________________________
SID:____________

- Closed book. One page, 2 sides of formula sheets. No calculators.

- There are 6 problems worth 100 points total.

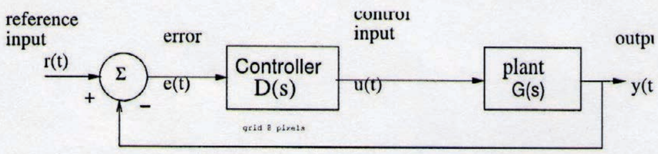
| Problem | Points | Score |
|---|---|---|
| 1 | 15 | |
| 2 | 16 | |
| 3 | 18 | |
| 4 | 20 | |
| 5 | 16 | |
| 6 | 15 | |
| Total | 100 | |

In the real world, unethical actions by engineers can cost money, careers, and lives. The penalty for unethical actions on this exam will be a grade of 'F' and a letter will be written for your file and to the Office of Student Conduct.

| | |
|---|---|
| $\tan^{-1}\frac{1}{10} = 5.7°$ | $\tan^{-1}\frac{1}{5} = 11.3°$ |
| $\tan^{-1}\frac{1}{4} = 14°$ | $\tan^{-1}\frac{1}{3} = 18.4°$ |
| $\tan^{-1}\frac{1}{2} = 26.6°$ | $\tan^{-1}\frac{1}{\sqrt{3}} = 30°$ |
| $\tan^{-1} 1 = 45°$ | $\tan^{-1}\sqrt{3} = 60°$ |
| $\sin 30° = \frac{1}{2}$ | $\cos 30° = \frac{\sqrt{3}}{2}$ |

| | | |
|---|---|---|
| $20\log_{10} 1 = 0dB$ | $20\log_{10} 2 = 6dB$ | $\pi \approx 3.14$ |
| $20\log_{10}\sqrt{2} = 3dB$ | $20\log_{10}\frac{1}{2} = -6dB$ | $2\pi \approx 6.28$ |
| $20\log_{10} 5 = 20db - 6dB = 14dB$ | $20\log_{10}\sqrt{10} = 10$ dB | $\pi/2 \approx 1.57$ |
| $1/e \approx 0.37$ | $\sqrt{10} \approx 3.164$ | $\pi/4 \approx 0.79$ |
| $1/e^2 \approx 0.14$ | $\sqrt{2} \approx 1.41$ | $\sqrt{3} \approx 1.73$ |
| $1/e^3 \approx 0.05$ | $1/\sqrt{2} \approx 0.71$ | $1/\sqrt{3} \approx 0.58$ |

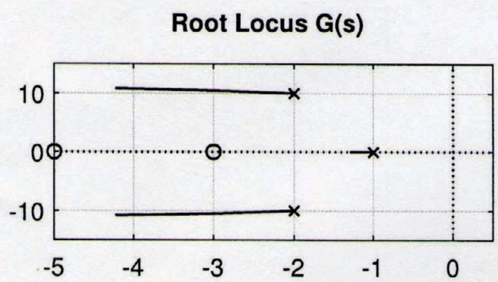
**Problem 1 (15 pts)**

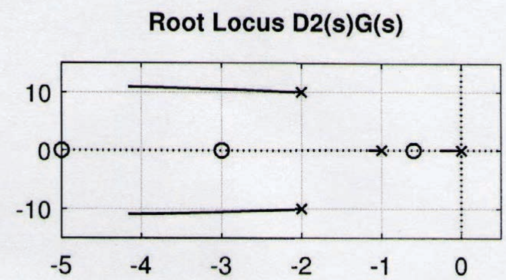


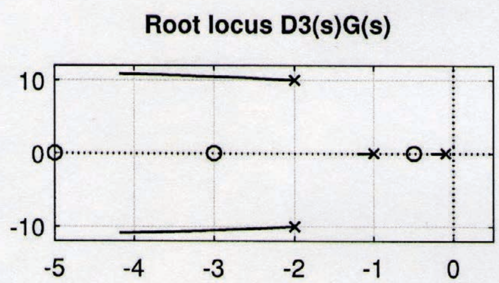

You are given the open-loop plant:

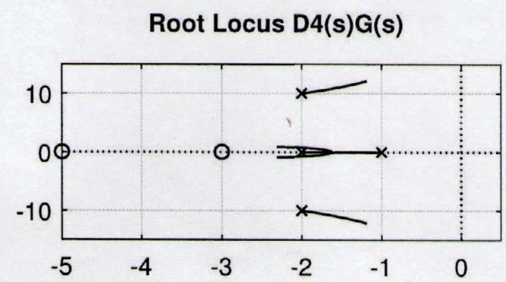
$$G(s) = \frac{5(s+5)(s+3)}{(s+1)(s^2+4s+104)}.$$

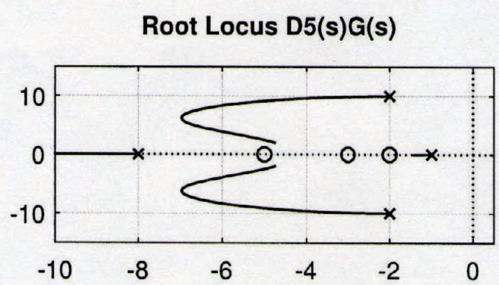
For the above system, the partial root locus is shown for 5 different controller/plant combinations, $G(s), D_2(s)G(s), ..., D_5(s)G(s)$. (Note: the root locus shows open-loop pole locations for $D(s)G(s)$, and closed-loop poles for $\frac{DG}{1+DG}$ are at end points of branches).




[5 pts] a) For each set of open-loop poles and zeros given above, choose the best corresponding open-loop Bode plot V,W,X,Y, or Z from the next page:

(i) $G(s)$: Bode Plot Y

(ii) $D_2(s)G(s)$: Bode plot X

(iii) $D_3(s)G(s)$: Bode plot W

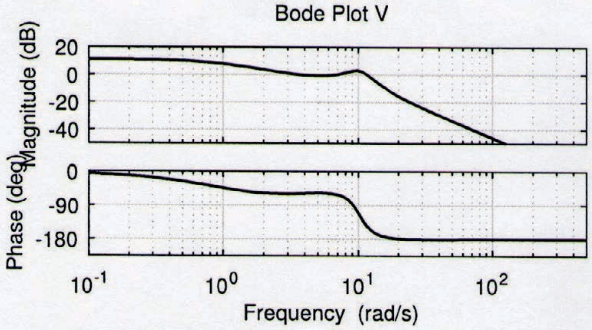
(iv) $D_4(s)G(s)$: Bode Plot V

(v) $D_5(s)G(s)$: Bode Plot Z

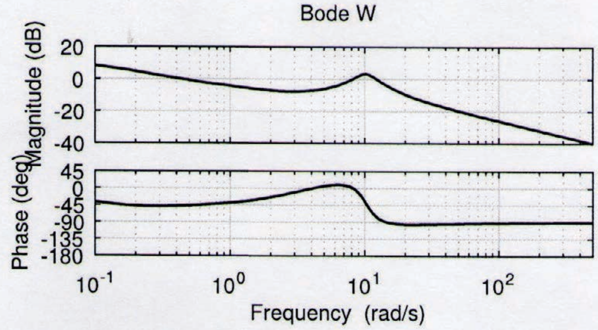
**Problem 1, cont.**

The open-loop Bode plots for 5 different controller/plant combinations, $D_1(s)G(s), ..., D_5(s)G(s)$ are shown below.

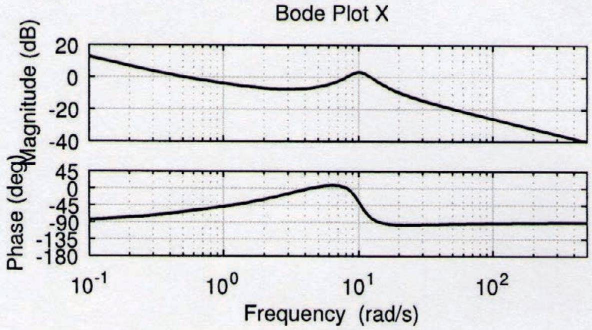
Bode Plot V



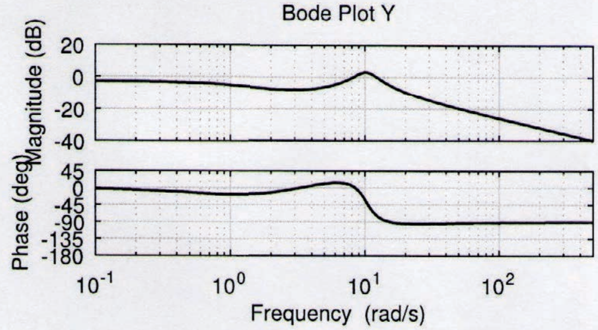

Bode W



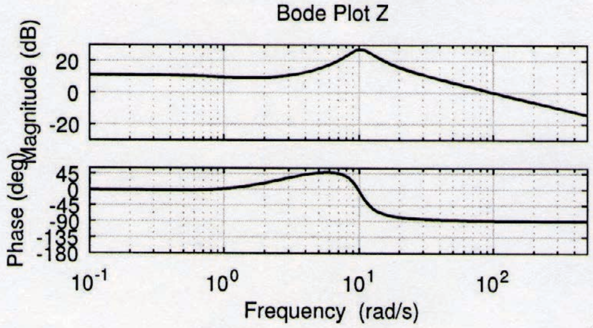

Bode Plot X




Bode Plot Y




Bode Plot Z




[5 pts] b) For the Bode plots above:

(i) Bode plot V: phase margin 40 (degrees) at $\omega =$ 11 rad/s
Bode plot V: gain margin 20 dB at $\omega =$ 23 rad/s
Estimate damping factor $\zeta =$ .3

(ii) Bode plot Z: phase margin 90 (degrees) at $\omega =$ 100
Bode plot Z: gain margin $\infty$ dB at $\omega =$ —

**Problem 1, cont.**

[5 pts] c) For each closed loop controller/plant with root locus as given in part a), choose the best corresponding closed-loop step response (A-E). (Note: dashed line shows final value.)

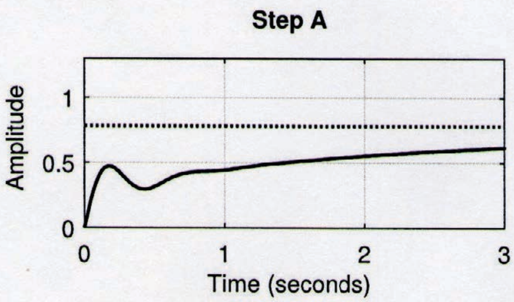
(i) $G(s)$: step response E

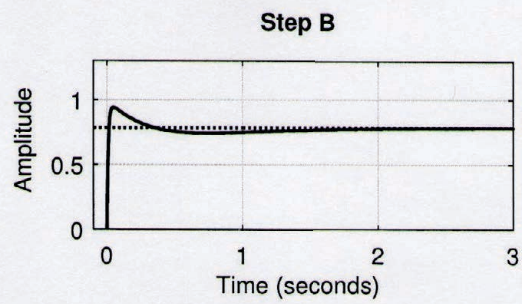
(ii) $D_2(s)G(s)$: step response D

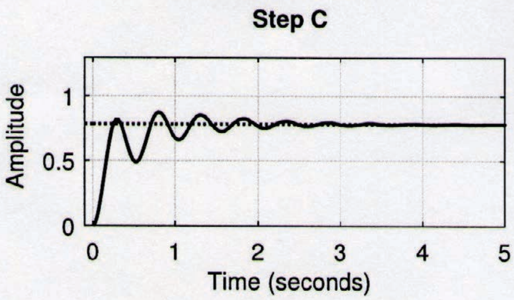
(iii) $D_3(s)G(s)$: step response A

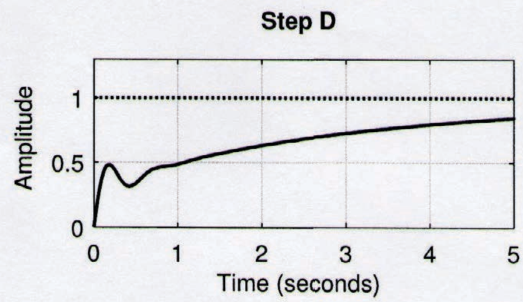
(iv) $D_4(s)G(s)$: step response C

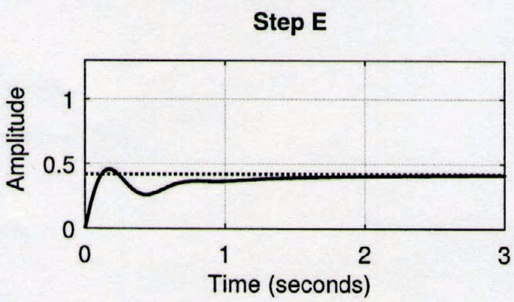
(v) $D_5(s)G(s)$: step response B



**Problem 2 (16 pts)**

The open-loop system is given by $G(s) = \frac{400}{(s+2)^2(s^2+2s+101)}$, and Bode plot for $G(s)$ is here:

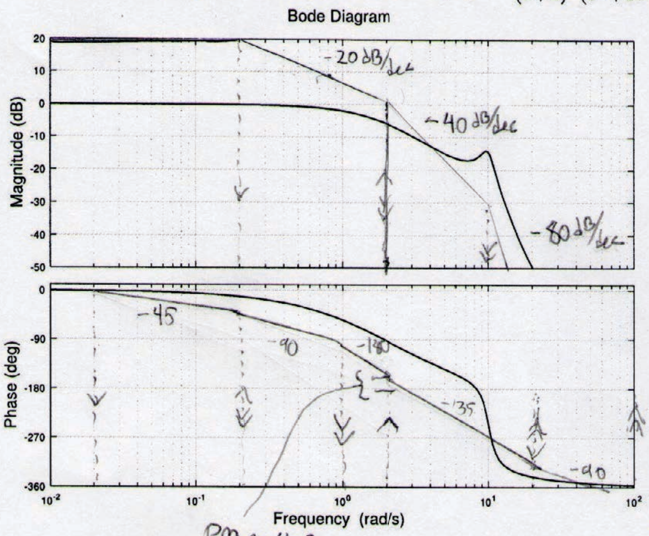



Fig. 3.1

PM ≈ 40°

A lag controller $D(s) = k\frac{s+\alpha}{s+\beta}$ is to be designed such that the unity gain feedback system with openloop transfer function $D(s)G(s)$ has static error constant $K_p = 10$. $D(s)G(s)$ should have a nominal (asymptotic approximation) phase margin $\phi_m \approx 40°$ at $\omega_{pm} = 2$ rad $s^{-1}$.

[6 pts] a. Determine gain, zero, and pole location for the lag network $D(s)$:

gain $k =$ 1 zero: $\alpha =$ 2 pole: $\beta =$ .2

[4 pts] b. Sketch the asymptotic Bode plot for the lag network $D(s)$ alone on the plot below:

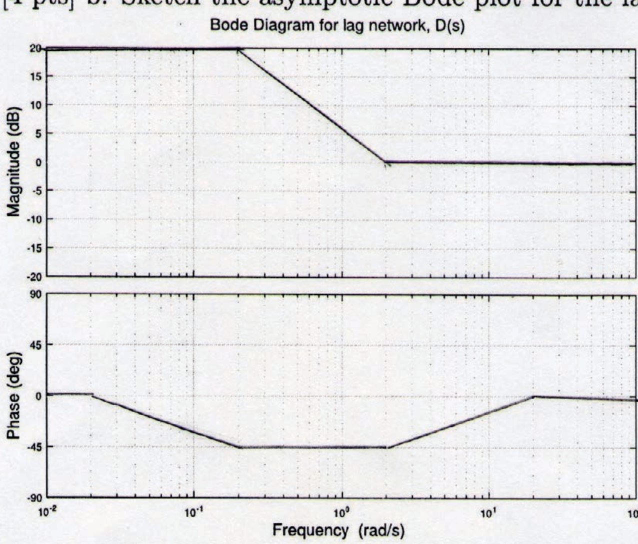



Fig. 3.2

[4 pts] c. Sketch the asymptotic Bode plot for the combined lag network and plant $D(s)G(s)$ on the plot (Fig. 3.1) at top of page.

[2 pts] d. Mark the phase margin and phase margin frequency on the plot of $D(s)G(s)$ (Fig. 3.1).

**Problem 3 (18 pts)**

[2 pt] a. Given the homogeneous linear differential equation $\dot{\mathbf{x}} = A\mathbf{x}$ with initial condition $\mathbf{x}(0) = \mathbf{x}_o$. Show that the solution $\mathbf{x}(t) = e^{At}\mathbf{x}_o$ satisfies both conditions.

$$\frac{d}{dt}(x(t)) = Ae^{At}x_0 = Ax(t) \checkmark$$

$$\lim_{t\to 0} x(t) = \lim_{t \to 0} e^{At}x_0 = x_0 \checkmark$$

dynamics

initial condition

[2 pt] b. Show that $e^{At}$ must equal $\mathcal{L}^{-1}[sI - A]^{-1}$. (Hint: see part a. above.)

$$\dot{x} = Ax$$

$$\mathcal{L}\{\dot{x} = Ax\} = \quad sIX(s) - X_0 = AX(s)$$

$$(sI - A)X(s) = X_0$$

$$x(t) = \mathcal{L}^{-1}\{(sI-A)^{-1}\}x_0$$

[2 pts] c. Given $\bar{A} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, find $e^{\bar{A}t}$

$$e^{\bar{A}t} = \left[\begin{array}{c|c} e^{\lambda_1 t} & 0 \\ \hline 0 & e^{\lambda_2 t} \end{array}\right]$$

[4 pts] d. Given $\bar{A}, A, P$ such that $\bar{A} = P^{-1}AP$ is diagonal, and given $e^{\bar{A}t}$. Also given the state vector $x = P\bar{x}$. Show how to find $e^{At}$ given $\bar{A}, A, P, e^{\bar{A}t}$, starting from $\dot{\bar{x}} = \bar{A}\bar{x}$. (Leave in general form.)

$e^{At} =$ $Pe^{\bar{A}t}P^{-1}$

$$\dot{\bar{x}} = \bar{A}\bar{x}$$

$$\bar{x} = e^{\bar{A}t}\bar{x}_0$$

$$P^{-1}x = e^{\bar{A}t}P^{-1}x_0$$

$$x = Pe^{\bar{A}t}P^{-1}x_0$$

from above

$$x = \exp(At)x_0$$

$$\Rightarrow e^{At} = Pe^{\bar{A}t}P^{-1}$$

**Problem 3, cont.**

Given the two LTI systems

$$\dot{\mathbf{x}}(t) = A\mathbf{x} + Bu = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u, \quad y = C\mathbf{x} = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\dot{\mathbf{z}}(t) = A_z\mathbf{z} + B_z u = \begin{bmatrix} -3 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} u, \quad y = C_z\mathbf{z} = [0 \quad 1] \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

[4 pts] e. Find a transformation $P$ such that $A = P^{-1}A_zP$ is diagonal. (Hint: this could be found using the controllability matrix for each system.)

$$\mathcal{C}_x = \begin{bmatrix} 1 & -1 \\ 2 & -6 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\mathcal{C}_z = \begin{bmatrix} -1 & 5 \\ 1 & -1 \end{bmatrix}$$

$$P = \mathcal{C}_z \mathcal{C}_x^{-1}$$

$$\hookrightarrow \mathcal{C}_x^{-1} = \frac{-1}{4} \begin{bmatrix} -6 & 1 \\ -2 & 1 \end{bmatrix}$$

$$P = \frac{-1}{4} \begin{bmatrix} -1 & 5 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -6 & 1 \\ -2 & 1 \end{bmatrix}$$

$$= -1/4 \begin{bmatrix} -4 & 4 \\ -4 & 0 \end{bmatrix}$$

[4 pts] f. Show that both systems have the same input-output behavior. That is, for the same input $u(t)$, the output $y(t)$ will be identical for both systems. Use $P$ from part e, and also verify $B_z$ and $C_z$ are correct.

note $z = Px$

$$P^{-1} = \frac{1}{2} \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$$

$$y = C_z z$$

$$y = C_z P x$$

$$= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} x$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x \checkmark$$

$$\dot{z} = A_z z + B_z u$$

$$P\dot{x} = A_z P x + B_z u$$

$$\dot{x} = P^{-1} A_z P x + P^{-1} B_z u$$

$$= Ax + \frac{1}{2} \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} u$$

$$\dot{x} = Ax + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u \checkmark$$

**Problem 4. (20 pts)**

Given the LTI system

$$\dot{\mathbf{x}}(t) = A\mathbf{x} + Bu = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \qquad y = C\mathbf{x} = [1 \quad 0]\mathbf{x},$$

[3 pts] a. Find $\mathbf{k} = [k_1 \ k_2]$ such that with state feedback $u = r - \mathbf{kx}$, the closed-loop poles of the system are at $\lambda_1, \lambda_2$.

$\Delta^{des}(s) = (s-\lambda_1)(s-\lambda_2)$

$= s^2 - (\lambda_1+\lambda_2)s + \lambda_1\lambda_2$

$k_1 =$ $\lambda_1\lambda_2$ $\quad k_2 =$ $-(\lambda_1+\lambda_2)$

$\Delta^{cl}(s) = \begin{vmatrix} s & -1 \\ +k_1 & s+k_2 \end{vmatrix}$

$= s^2 + k_2 s - k_1$

[1 pts] b. The initial condition is $\mathbf{x}(0) = [0 \ \ 0]^T$. For $r(t)$ a unit step input, it is required that $x_1(t) < 1 \ \forall t$, that is over shoot is not allowed.

What is range of $\lambda_1, \lambda_2$ to avoid over shoot?

$\lambda_i < 0, \quad \lambda_i \in \mathbb{R}$

[3 pts] c. Assume $\mathbf{k} = [k_1 \ k_2] = [4 \ \ 5]$. Let $e(t) = r(t) - C\mathbf{x}$. For $r(t)$ a unit step input, find the steady state error.

$\lim_{t\to\infty} e(t) =$ $3/4$

$e_{ss} = 1 + C A_{cl}^{-1} B$

$A_{cl} = \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix} \qquad A_{cl}^{-1} = \frac{1}{4}\begin{bmatrix} -5 & -1 \\ 4 & 0 \end{bmatrix}$

$e_{ss} = 1 + [1 \ \ 0] \frac{1}{4} \begin{bmatrix} -5 & -1 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

[3 pts] d. For $\mathbf{k} = [k_1 \ k_2] = [4 \ \ 5]$, with $u = r - \mathbf{kx}$, find $\frac{Y(s)}{R(s)}$. (Express the transfer function as a ratio of polynomials, not as matrix operations.)

$\frac{Y(s)}{R(s)} = \dfrac{1}{s^2+5s+4}$

$\frac{Y(s)}{R(s)} = C(sI - A_{cl})^{-1}B$

$(sI - A_{cl})^{-1} = \frac{1}{\Delta^{cl}(s)} \begin{bmatrix} s+5 & 1 \\ -4 & s \end{bmatrix}$

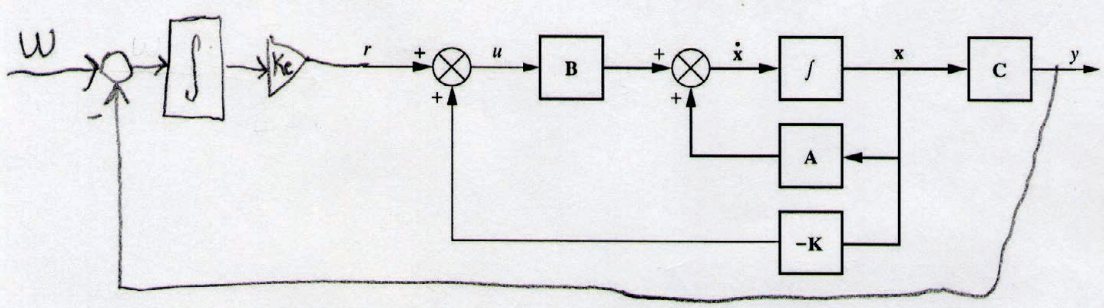
$C(sI-A_{cl})^{-1}B = \dfrac{1}{s^2+5s+4}$

**Problem 4, cont. (20 pts)**

[4 pts] e. Define $e_w(t)$ to be the error between an input $w(t)$ and output $y(t)$. That is, $e_w(t) = w(t) - y(t)$. We desire to find an input $r(w, y)$ to the state feedback system shown below in part f such that $\lim_{t\to\infty} e_w(t) = 0$ for a step input $w(t)$ of any amplitude.

$r(w, y) = k_e \int_0^t (w(\tau) - y(\tau))\,d\tau$

[2 pts] f. Using the controller from part e, expand the block diagram below to include the controller and input $w$.




[4 pts] g. Assume the overall control system is stable, and refer to the expanded block diagram above. Describe in words why $\lim_{t\to\infty} e_w(t) = 0$ for a step input $w(t)$.

Suppose a constant error $e_w(t) \neq 0$. Over time r will grow as this error accumulates, increasing the resulting reference passed to state fb system, driving the output higher or lower. $Y(t)$ as $t \to \infty$ must be bounded and constant (stability shows this), thus contradiction

$\Rightarrow e_w(t) = 0$

**Problem 5. 13 pts**
Given the following system model:

$$\dot{\mathbf{x}} = A\mathbf{x} + Bu = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -k_2 & 1 \\ 0 & -k_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \qquad y = C\mathbf{x} = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

[2 pts] a. Determine if the system $A, B, C$ is controllable, and restrictions if any on $k_1, k_2$ for controllability.

$$\mathcal{C} = [B \mid AB] = \begin{bmatrix} 0 & 1 \\ 1 & -k_1 \end{bmatrix}$$

$\therefore$ Completely Controllable $\forall\, k_1, k_2 \in \mathbb{R}$

[2 pts] b. Determine if the system $A, B, C$ is observable, and restrictions if any on $k_1, k_2$ for observability.

$$\mathcal{O} = \left[\frac{C}{CA}\right] = \begin{bmatrix} 1 & 0 \\ -k_2 & 1 \end{bmatrix}$$

$\therefore$ Completely Observable $\forall\, k_1, k_2 \in \mathbb{R}$

[2 pts] c. Provide state equations for an observer which takes as inputs $u(t), y(t)$, and provides an estimate of the state $\hat{\mathbf{x}}(t)$.

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y})$$
$$\hat{y} = C\hat{x}$$

[6 pts] d. Given $k_1 = 1, k_2 = 4$, find observer gain $L$ such that the observer has closed loop poles at $s_1 = -10, s_2 = -10$.

$$L = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} = \begin{bmatrix} 15 \\ 81 \end{bmatrix} =$$

$$\bar{A} = A - LC = \begin{bmatrix} -4 & 1 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} [1 \quad 0] = \begin{bmatrix} -l_1 - 4 & 1 \\ -l_2 & -1 \end{bmatrix}$$

$$|sI - \bar{A}| = (s + (4 + l_1))(s+1) + l_2 = s^2 + (l_1 + 5)s + (l_1 + l_2 + 4)$$

$$(s+10)(s+10) = s^2 + 20s + 100$$

$\therefore$ $l_1 = 15$ $\quad (l_1 + 5 = 20)$

$l_2 = 81$ $\quad (l_1 + l_2 + 4 = 100)$

**Problem 5, cont.**

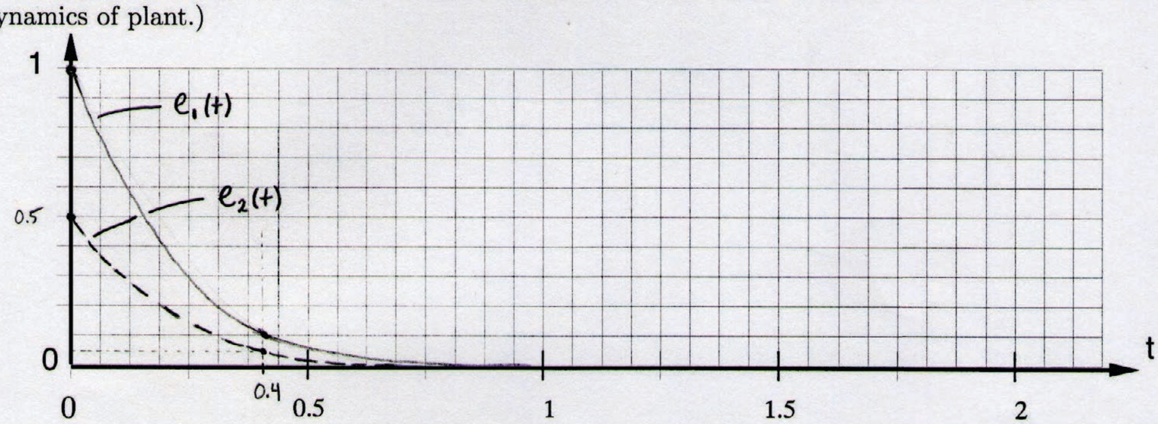
[4 pts] e. Let the error between the estimated state and the true state be given by $\mathbf{e}(t) = \hat{\mathbf{x}} - \mathbf{x}$. Find the dynamics of the error in terms of $A, B, C, L$.

$\dot{\mathbf{e}} =$ $\underline{(A - LC)\,\underline{e}}$

[4 pts] f. Given initial conditions

$$\mathbf{x} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} \text{ and } \hat{\mathbf{x}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Sketch approximately $e_1(t), e_2(t)$ for $t \geq 0$. (Hint: consider dynamics of observer compared to dynamics of plant.)




<u>Given:</u>

$\dot{\underline{e}} = (A - LC)\,\underline{e}$ with closed loop poles @ $s = -10, -10$

↳ Critically damped

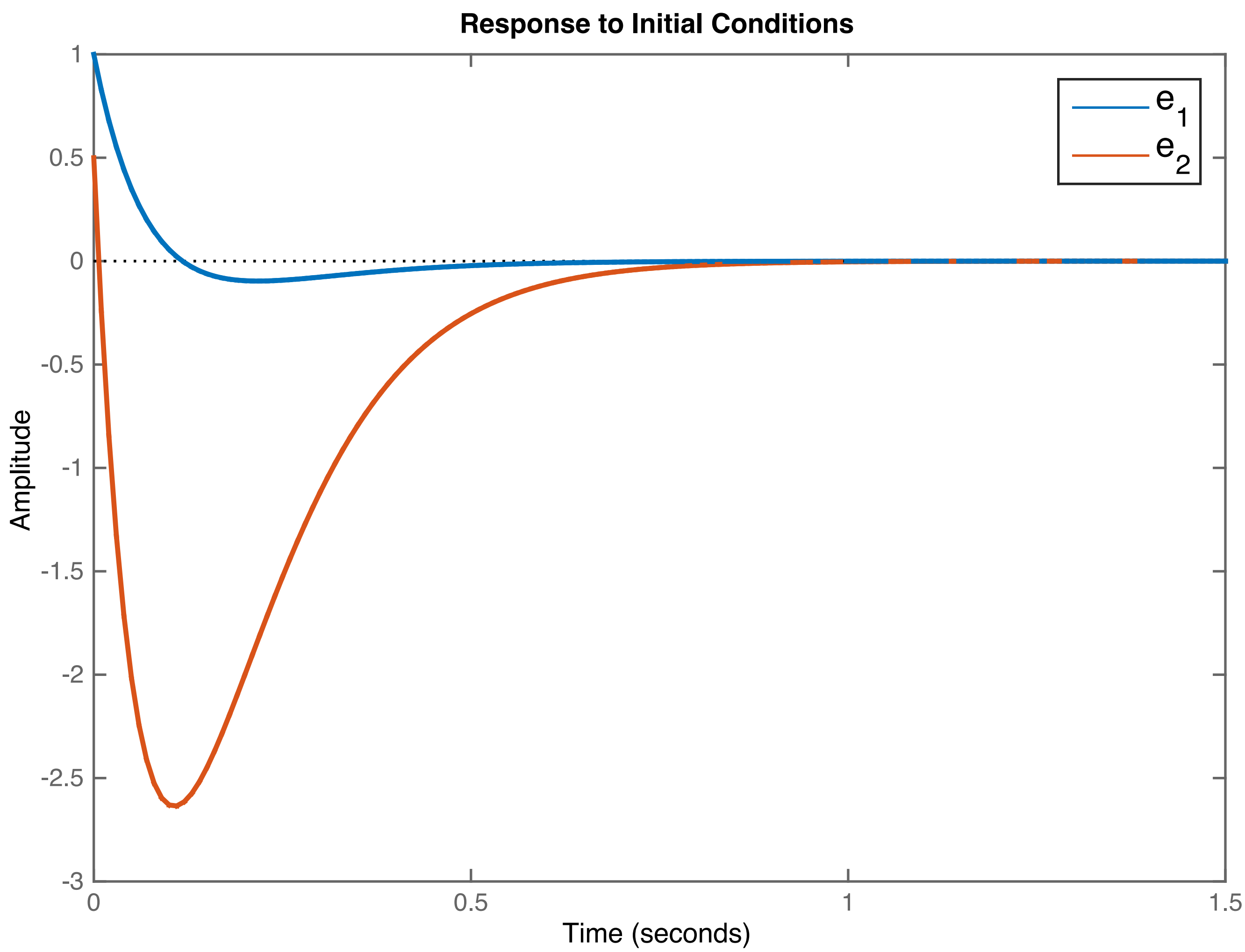
↳ $e(0) = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$

$T_s \cong 4/\sigma_D = 4/10 = 0.4$ sec to 90% settle

(Dynamics of $\underline{e}(t)$ are independent from plant dynamics by separability)

Accepted for Full Credit...

True response has zeros and the initial response is on the next page.


Response to Initial Conditions
Amplitude
Time (seconds)
e1
e2
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
-3
0
0.5
1
1.5

**Problem 6 (8 pts)**

[4 pts] a. Given the discrete time system below, find $\mathbf{X}(z)$ the z-transform of $\mathbf{x}(k)$, where $u(k) = (\frac{1}{2})^k$ for $k \geq 0$.

$$\mathbf{x}(k+1) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

$\mathbf{X}(z) = \begin{bmatrix} 1/(z)(z-1/2) \\ 1/(z-1/2) \end{bmatrix}$

$$X(z) = (zI - G)^{-1} (H\,U(z) + x[0]\,z) \text{ , assuming } x[0] = 0$$

$$= (zI - G)^{-1} (H)\left(\frac{z}{z-1/2}\right)$$

$$= \begin{bmatrix} z & -1 \\ 0 & z \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ z/(z-1/2) \end{bmatrix}$$

$$= \begin{bmatrix} 1/z & 1/z^2 \\ 0 & 1/z \end{bmatrix} \begin{bmatrix} 0 \\ z/(z-1/2) \end{bmatrix} \Rightarrow \begin{bmatrix} 1/(z)(z-1/2) \\ 1/(z-1/2) \end{bmatrix}$$

[4 pts] b. Given

$$\mathbf{x}(k+1) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

Determine the response of the system to $u(k)$ a unit step input.

$\mathbf{x}(k) = \begin{bmatrix} u(k-2) \\ u(k-1) \end{bmatrix}$

Using above ...

$$X(z) = \begin{bmatrix} 1/z & 1/z^2 \\ 0 & 1/z \end{bmatrix} \begin{bmatrix} 0 \\ z/(z-1) \end{bmatrix} \Rightarrow \begin{bmatrix} (z^{-2})(z/(z-1)) \\ (z^{-1})(z/(z-1)) \end{bmatrix}$$

Inverse Z-Transform yields a 1k and 2k delay on the unit step...

$$x(k) = \begin{bmatrix} u(k-2) \\ u(k-1) \end{bmatrix}$$

[4 pts] c. Given

$$X(z) = \frac{z^2}{(z-\frac{1}{2})(z-1)}$$

find $x(k)$ for $k \geq 0$.

$x(k) = \underline{-(1/2)^k + 2u(k)}$

$$\frac{X(z)}{z} = \frac{z}{(z-1/2)(z-1)} = \frac{A}{(z-1/2)} + \frac{B}{(z-1)}$$

$$z = A(z-1) + B(z-1/2)$$

$\hookrightarrow z = 1: \quad 1 = B(1/2) \Rightarrow B = 2$

$\hookrightarrow z = 1/2: \quad 1/2 = A(-1/2) \Rightarrow A = -1$

$$X(z) = \frac{-z}{(z-1/2)} + \frac{2z}{(z-1)}$$

$$\Downarrow$$

$$x(k) = -(1/2)^k + 2u(k)$$

**Problem 6, cont.)**

[4 pts] d. Given

$$X(z) = \frac{z}{(z-\frac{1}{2})(z-1)(z-\frac{2}{5})}$$

find $\lim_{k\to\infty} x(k) =$ 10/3

Final Value Thm:
$$\lim_{k\to\infty} x(k) = \lim_{z\to 1} \left(\frac{(z-1)}{z}\right) X(z)$$
$$= \lim_{z\to 1} \frac{\cancel{(z-1)}\cancel{(z)}}{\cancel{(z)}(z-1/2)\cancel{(z-1)}(z-2/5)}$$
$$= 1/((1/2)(3/5)) = 10/3$$

[4 pts] e. Given a mass $m$, and input force $f$, $\ddot{x} = f/m$. Let the state $x_1$ be the position and $x_2$ velocity of the mass. The continuous time state equations for the system are :

$$\dot{\mathbf{x}} = A\mathbf{x} + Bf = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} f(t),$$

Find the discrete time equivalent system using zero-order hold for input force $f(t)$ and sampling period $T$: $\mathbf{x}((k+1)T) = G\mathbf{x}(kT) + Hf(kT)$.

$$G = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \qquad H = \begin{bmatrix} T^2/2m \\ T/m \end{bmatrix}$$

$G = e^{AT}$

$$e^{At} = \mathcal{L}^{-1}\{(sI-A)^{-1}\}$$
$$= \mathcal{L}^{-1}\left\{\begin{bmatrix} s & -1 \\ 0 & s \end{bmatrix}^{-1}\right\}$$
$$= \mathcal{L}^{-1}\left\{\begin{bmatrix} 1/s & 1/s^2 \\ 0 & 1/s \end{bmatrix}\right\}$$
$$e^{At} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$
$$\Downarrow$$
$$e^{AT} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}$$

$$H = \left(\int_0^T e^{A\lambda} d\lambda\right) B$$
$$= \left(\int_0^T \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1/m \end{bmatrix} d\lambda\right)$$
$$= \int_0^T \begin{bmatrix} \lambda/m \\ 1/m \end{bmatrix} d\lambda$$
$$= \begin{bmatrix} \lambda^2/2m \\ \lambda/m \end{bmatrix}\Bigg|_0^T$$
$$H = \begin{bmatrix} (\frac{1}{2m})T^2 \\ (\frac{1}{m})T \end{bmatrix}$$