First Midterm Examination
Monday September 29 2014
Closed Books and Closed Notes
Answer Both Questions

Question 1
A Particle on a Spinning Cone
20 Points

As shown in Figure 1, a particle of mass m is attached to a fixed point O by a linearly elastic spring. The spring has a stiffness K and an unstretched length ℓ_0. The particle is free to move on the smooth surface of a cone which rotates about the vertical z axis with a speed $\Omega(t)$. A vertical gravitational force $-mg\mathbf{E}_3$ acts on the particle, and the semi angle $\frac{\pi}{2} - \alpha$ of the cone is constant.

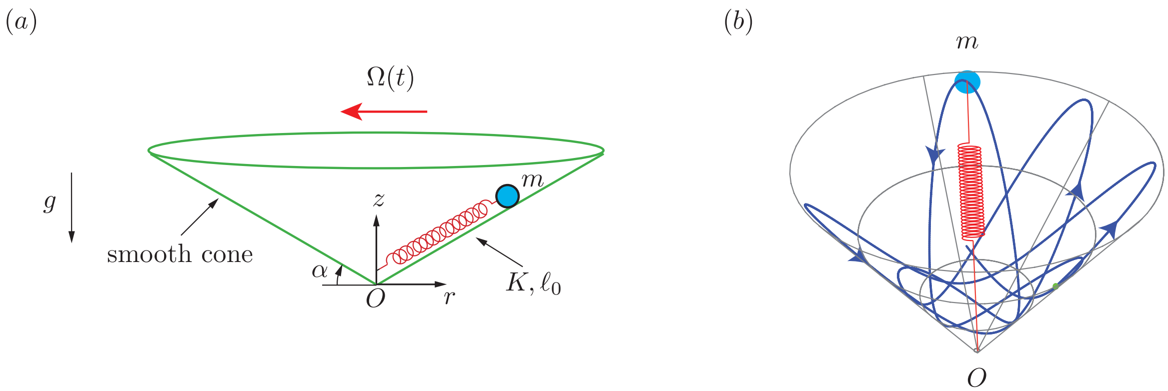


Figure 1: *(a) Schematic of a particle of mass m which is attached to a fixed point O by an elastic spring. A vertical gravitational force $-mg\mathbf{E}_3$ acts on the particle and the particle is free to move in a smooth cone which is rotating about the vertical z axis with a non-constant speed $\Omega = \Omega(t)$. (b) Representative motion of the particle on the cone.*

In your answers to the questions below, please make use of the results on spherical polar coordinates on Page 3.

(a) *(5 Points)* What is the constraint on the motion of the particle? Give a prescription for the constraint force enforcing this constraint.

(b) *(5 Points)* Draw a freebody diagram of the particle. Your freebody diagram should include a clear expression for the spring force.

(c) *(5 Points)* Establish the second-order differential equations governing the motion of the particle.

(d) *(5 Points)* Show that the total energy E and the angular momentum $\mathbf{H}_O \cdot \mathbf{E}_3$ of the particle are conserved during the motion of the particle.

Question 2

A Particle on a Surface

30 Points

As shown in Figure 2, a particle of mass m is free to move on a surface $z = f(x)$.

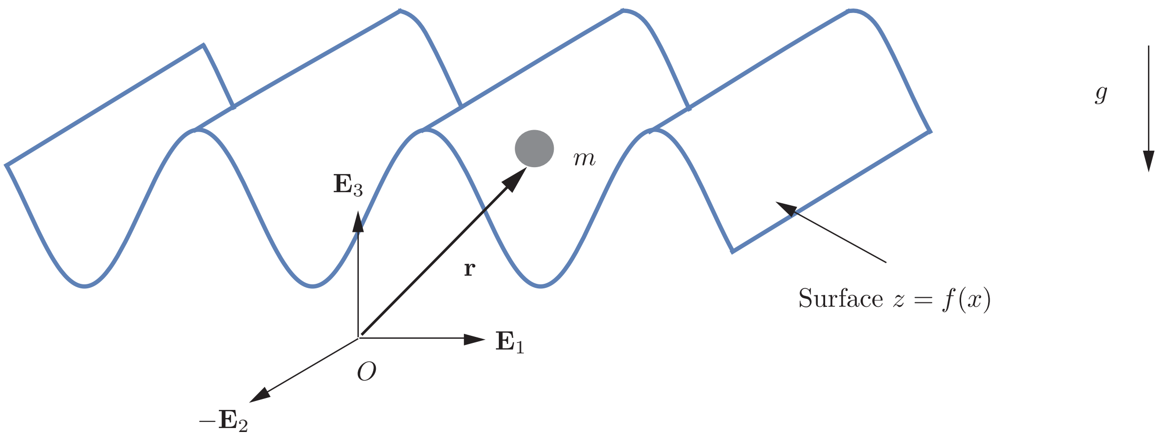


Figure 2: *Schematic of a particle of mass m which is moving on a rough surface $z = f(x)$ in $\mathbb{E}^3$ under the influence of a gravitational force.*

To establish the equations of motion for the particle, the following curvilinear coordinate system is defined for $\mathbb{E}^3$:

$$q^1 = x, \qquad q^2 = y, \qquad q^3 = \eta = z - f(x). \tag{1}$$

(**a**) *(7 Points)* Show that the covariant basis vectors for this system are

$$\mathbf{a}_1 = \mathbf{E}_1 + \frac{\partial f}{\partial x}\mathbf{E}_3, \qquad \mathbf{a}_2 = \mathbf{E}_2, \qquad \mathbf{a}_3 = \mathbf{E}_3. \tag{2}$$

Compute the matrix $[a_{ik}]$. You will find it helpful to use the abbreviation $f_x = \frac{\partial f}{\partial x}$.

(**b**) *(8 Points)* What are the contravariant basis vectors $\mathbf{a}^k$ for this coordinate system? Compute the inverse of the matrix $[a_{ik}]$.

(**c**) *(10 Points)* Assuming the particle is in motion on the rough surface $z = f(x)$ under a gravitational force $-mg\mathbf{E}_3$, establish the equations of motion for the particle.

(**d**) *(5 Points)* Show that the equations of motion of a particle *constrained* to move on a smooth q^1 coordinate curve in the presence of a gravitational force $-mg\mathbf{E}_3$ can be expressed in the form

$$m\left(1 + f_x^2\right)\ddot{x} + m f_x f_{xx}\dot{x}^2 \quad = \quad -mg f_x, \tag{3}$$

where $f_{xx} = \frac{\partial^2 f}{\partial x \partial x}$ Using the equation of motion (3), compute possible equilibrium positions of the particle and give a physical interpretation of the positions you find. Feel free to use the specific example $f(x) = A\sin\left(\frac{\pi x}{\ell}\right)$ to illustrate your answer if you wish.

Notes on Spherical Polar Coordinates

Recall that the spherical polar coordinates $\{R, \phi, \theta\}$ are defined using Cartesian coordinates $\{x = x_1, y = x_2, z = x_3\}$ by the relations:

$$R = \sqrt{x_1^2 + x_2^2 + x_3^2}\,, \quad \theta = \arctan\left(\frac{x_2}{x_1}\right)\,, \quad \phi = \arctan\left(\frac{\sqrt{x_1^2 + x_2^2}}{x_3}\right).$$

In addition, it is convenient to define the following orthonormal basis vectors:

$$\begin{bmatrix} \mathbf{e}_R \\ \mathbf{e}_\phi \\ \mathbf{e}_\theta \end{bmatrix} = \begin{bmatrix} \cos(\theta)\sin(\phi) & \sin(\theta)\sin(\phi) & \cos(\phi) \\ \cos(\theta)\cos(\phi) & \sin(\theta)\cos(\phi) & -\sin(\phi) \\ -\sin(\theta) & \cos(\theta) & 0 \end{bmatrix} \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{bmatrix}.$$

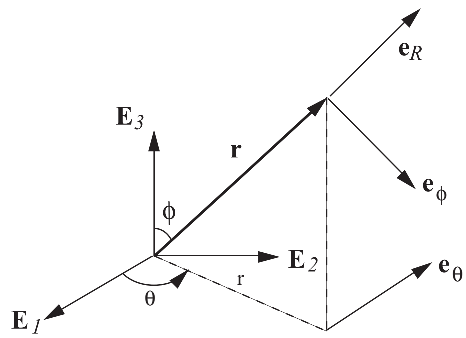


Figure 3: Spherical polar coordinates

For the coordinate system $\{R, \phi, \theta\}$, the covariant basis vectors are

$$\mathbf{a}_1 = \mathbf{e}_R\,, \quad \mathbf{a}_2 = R\mathbf{e}_\phi\,, \quad \mathbf{a}_3 = R\sin(\phi)\mathbf{e}_\theta.$$

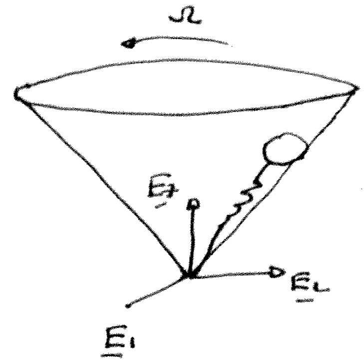
In addition, the contravariant basis vectors are

$$\mathbf{a}^1 = \mathbf{e}_R\,, \quad \mathbf{a}^2 = \frac{1}{R}\mathbf{e}_\phi\,, \quad \mathbf{a}^3 = \frac{1}{R\sin(\phi)}\mathbf{e}_\theta.$$

For a particle of mass m which is unconstrained, the linear momentum $\mathbf{G}$, angular momentum $\mathbf{H}_O$ and kinetic energy T of the particle are

$$\begin{aligned} \mathbf{G} &= m\dot{R}\mathbf{a}_1 + m\dot{\phi}\mathbf{a}_2 + m\dot{\theta}\mathbf{a}_3\,, \\ \mathbf{H}_O &= mR^2\left(\dot{\phi}\mathbf{e}_\theta - \dot{\theta}\sin(\phi)\mathbf{e}_\phi\right)\,, \\ T &= \frac{m}{2}\left(\dot{R}^2 + R^2\dot{\phi}^2 + R^2\sin^2(\phi)\dot{\theta}^2\right). \end{aligned}$$

QUESTION 1



Use Spherical Polar Cordinate System:

$$q^1 = R, \quad q^2 = \theta, \quad q^3 = \phi$$

(a)

Constraint $\qquad \phi + \alpha - \pi/2 = 0$

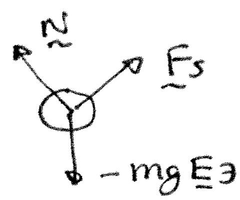
$$\Leftrightarrow \quad \underline{v} \cdot \frac{1}{R}\underline{e}_\phi = 0 \qquad (\underline{v} \cdot \underline{a}^3 = 0)$$

Constraint force

$$\underline{F}_c = \underline{N} = \frac{\lambda}{R}\underline{e}_\phi$$

Because the cone is smooth, the fact that it is spinning has no effect on the particle.

(b)



$$\underline{F}_s = -K(R-\ell_0)\underline{e}_R$$

(c) Can use approach II and Lagrangian

$$\tilde{L} = \frac{1}{2}m(\dot{R}^2 + R^2\dot{\theta}^2\sin^2\phi_0) - mgR\cos\phi_0 - \frac{1}{2}K(R-\ell_0)^2$$

Here $\qquad \phi_0 = \frac{\pi}{2} - \alpha \qquad \cos\phi_0 = \sin\alpha, \quad \sin\phi_0 = \cos\alpha$

Equations of motion

$$\frac{d}{dt}\left(\frac{\partial \tilde{L}}{\partial \dot{R}} = m\dot{R}\right) - \left(\frac{\partial \tilde{L}}{\partial R} = -mg\cos\phi_0 + mR\dot{\theta}^2\cos^2\alpha - K(R-\ell_0)\right) = 0$$

$$\frac{d}{dt}\left(\frac{\partial \tilde{L}}{\partial \dot{\phi}} = mR^2\dot{\theta}\cos^2\alpha\right) - \frac{\partial \tilde{L}}{\partial \theta} = 0.$$

In summary

$$m\ddot{R} + mg\sin\alpha - mR\dot{\theta}^2\cos^2\alpha + K(R-\ell_0) = 0$$

$$\frac{d}{dt}(mR^2\dot{\theta}\cos^2\alpha) = 0.$$

(d)

(i) $\dot{T} = \underline{F}.\underline{V} = \underline{F_c}.\underline{V} - mg\underline{E_3}.\underline{V} - K(R-\ell_0)\underline{E_R}.\underline{V}$

$$= 0 - \frac{d}{dt}\left(mg\underline{E_3}.\underline{r} + \frac{K}{2}(R-\ell_0)^2\right)$$

Hence $\frac{d}{dt}\left(T + mg\underline{E_3}.\underline{r} + \frac{K}{2}(R-\ell_0)^2 = E\right) = 0$

Note $E = \frac{1}{2}m(\dot{R}^2 + R^2\dot{\theta}^2\cos^2\alpha) + \frac{K}{2}(R-\ell_0)^2 - mgR\sin\alpha$

(ii)

$\underline{H_o}.\underline{E_3} = mR^2\dot{\theta}\sin^2\phi_0 = mR^2\dot{\theta}\cos^2\alpha$

From 2 Eom $\frac{d}{dt}\left(mR^2\dot{\theta}\cos^2\alpha\right) = 0 \Rightarrow \underline{H_o}.\underline{E_3}$ is conserved.

Alternatively

$$\dot{\underline{H_o}}\cdot\underline{E_3} = (\underline{r}\times\underline{F}).\underline{E_3}$$

$$= (\underline{r}\times -mg\underline{E_3} + \underline{r}\times\underline{F_s} + \underline{r}\times\underline{F_c}).\underline{E_3}$$

$$= 0 + 0 + 0$$

$\Rightarrow$ $\dot{\underline{H_o}}\cdot\underline{E_3}$ is conserved

$\Rightarrow$ $\frac{d}{dt}\left(\underline{H_o}.\underline{E_3}\right)$ is conserved.

Common Error:

The most common error made with this problem was to impose an additional constraint $\dot{\theta} = \Omega$ on the particle. If this constraint is imposed then E and $\underline{H_o}.\underline{E_3}$ are <u>not</u> conserved.

QUESTION 2

$q^1 = x \qquad q^2 = y \qquad q^3 = z - f(x)$

(a) $\underset{\sim}{r} = x\underset{\sim}{E}_1 + y\underset{\sim}{E}_2 + (q^3 + f)\underset{\sim}{E}_3$ where $f = f(x)$.

$\underset{\sim}{a}_1 = \dfrac{\partial \underset{\sim}{r}}{\partial x} = \underset{\sim}{E}_1 + f'\underset{\sim}{E}_3 \qquad f' = f_x = \dfrac{\partial f}{\partial x}$

$\underset{\sim}{a}_2 = \dfrac{\partial \underset{\sim}{r}}{\partial y} = \underset{\sim}{E}_2$

$\underset{\sim}{a}_3 = \dfrac{\partial \underset{\sim}{r}}{\partial q^3} = \underset{\sim}{E}_3$

$$[a_{ik}] = \begin{bmatrix} 1 + f_x^2 & 0 & f_x \\ 0 & 1 & 0 \\ f_x & 0 & 1 \end{bmatrix}$$

(b) $\underset{\sim}{a}^k = \nabla q^k$

$\underset{\sim}{a}^1 = \nabla x = \underset{\sim}{E}_1$

$\underset{\sim}{a}^2 = \nabla y = \underset{\sim}{E}_2$

$\underset{\sim}{a}^3 = \nabla q^3 = \underset{\sim}{E}_3 - f'\underset{\sim}{E}_1$

$$[a_{ik}]^{-1} = [a^{ik}] = \begin{bmatrix} 1 & 0 & -f_x \\ 0 & 1 & 0 \\ -f_x & 0 & 1 + f_x^2 \end{bmatrix}$$

Using the definition $a^k = \nabla q^k$ is the easiest method to determine the contravariant basis vectors.

(c)

$$T = \tfrac{1}{2} m \underline{v}\cdot\underline{v}$$
$$= \tfrac{1}{2} m \left(\dot{x}\,\underline{E}_1 + \dot{y}\,\underline{E}_2 + (\dot{n} + f_x \dot{x})\,\underline{E}_3 \right) \cdot \underline{v}$$
$$= \tfrac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 + (\dot{n} + \dot{x} f_x)^2 \right)$$

$$\underline{F} = -mg\,\underline{E}_3 + \lambda\,\underline{a}^3 - \mu_K \|\lambda \underline{a}^3\| \frac{\underline{v}_{rel}}{\|\underline{v}_{rel}\|}$$

$$\underline{v}_{rel} = \dot{x}\,\underline{a}_1 + \dot{y}\,\underline{a}_2$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}} = m\dot{x} + m(\dot{n} + \dot{x} f_x) f_x \right) - \left(\frac{\partial T}{\partial x} = m(\dot{n} + \dot{x} f_x)\,\dot{x} f_{xx} \right)$$
$$= \underline{F}\cdot\underline{a}_1 = -\mu_K \|\lambda \underline{a}^3\| \frac{\underline{v}_{rel}\cdot\underline{a}_1}{\|\underline{v}_{rel}\|}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}} = m\dot{y} \right) - \left(\frac{\partial T}{\partial y} = 0 \right) = \underline{F}\cdot\underline{a}_2 = -\mu_K \|\lambda \underline{a}^3\| \frac{\underline{v}_{rel}\cdot\underline{a}_2}{\|\underline{v}_{rel}\|}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{n}} = m(\dot{n} + \dot{x} f_x) \right) - \left(\frac{\partial T}{\partial n} = 0 \right) = \underline{F}\cdot\underline{a}_3$$
$$= -mg + \lambda - \mu_K \|\lambda \underline{a}^3\| \frac{\underline{v}_{rel}\cdot\underline{a}_3}{\|\underline{v}_{rel}\|}$$

Now introduce constraint $n = 0$:

$$\frac{d}{dt}\left(m(1 + f_x^2)\,\dot{x} \right) - m f_x f_{xx} \dot{x}^2 = -\mu_K \|\lambda \underline{a}^3\| \frac{\underline{v}_{rel}\cdot\tilde{\underline{a}}_1}{\|\underline{v}_{rel}\|}$$

$$m\ddot{y} = -\mu_K \|\lambda \underline{a}^3\| \frac{\underline{v}_{rel}\cdot\tilde{\underline{a}}_2}{\|\underline{v}_{rel}\|}$$

$$m(\ddot{x} f_x + \dot{x}^2 f_{xx}) = -mg + \lambda - \mu_K \|\lambda \underline{a}^3\| \frac{\underline{v}_{rel}\cdot\tilde{\underline{a}}_3}{\|\underline{v}_{rel}\|}$$

(d) For a particle moving on a smooth q' c.c. we can use approach I

$$\tilde{T} = \frac{m}{2}\left(\dot{x}^2(1+f_x^2)\right)$$

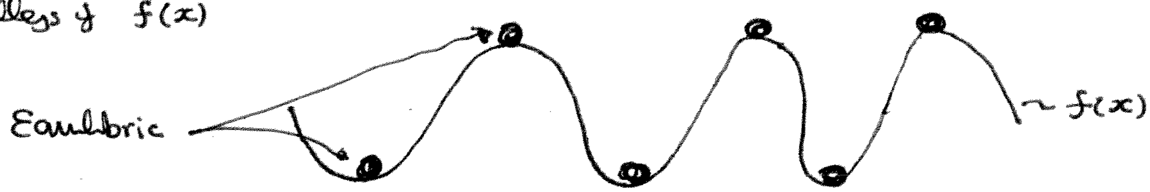
$$\tilde{U} = mg f(x) \qquad \underset{\sim}{F_c} = \lambda_1 \underline{a}^2 + \lambda_2 \underline{a}^3$$

$$\frac{d}{dt}\left(\frac{\partial \tilde{L}}{\partial \dot{x}} = m\dot{x}(1+f_x^2)\right) - \left(m\dot{x}^2 f_x f_{xx} - mg f_x\right) = 0$$

Hence expanding:

$$m(1+f_x^2)\ddot{x} + m\dot{x}^2 f_x f_{xx} = -mg f_x$$

Equilibria occur when $\dot{x} = \ddot{x} = 0 \Rightarrow f_x = 0$ i.e. at the peaks and valleys of $f(x)$



Common Errors:

The main error with this problem was using T correctly. Some students erroneously used

$$T = \tfrac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

for (c). Others, used Approach II. Because of friction, Approach II doesn't give the equations of motion.