First Midterm Examination
Thursday September 28 2011
Closed Books and Closed Notes
Answer All Three Questions

Question 1
A Particle in a Newtonian Gravitational Field
25 Points

As shown in Figure 1, a particle of mass m is free to move in space. The particle is under the influence of a gravitational force

$$\mathbf{F}_n = -\frac{GMm}{\|\mathbf{r}\|^3}\mathbf{r} \tag{1}$$

where $\mathbf{r}$ is the position vector of the particle relative to a fixed point O.

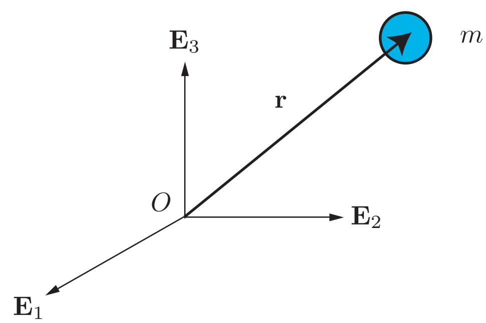


Figure 1: *Schematic of a particle of mass m which is moving in $\mathbb{E}^3$ under the influence of a gravitational force $\mathbf{F}_n$.*

In your answers to the questions below, please make use of the results on spherical polar coordinates on Page 4.

(**a**) *(5 Points)* Using a spherical polar coordinate system, give an expression for the Lagrangian $L = T - U$ of the particle.

(**b**) *(5 Points)* What are Lagrange's equations of motion for the particle?

(**c**) *(5 Points)* Starting from the work energy theorem $\dot{T} = \mathbf{F} \cdot \mathbf{v}$ show that the total energy E of the particle is conserved.

(**d**) *(3 Points)* Using the angular momentum theorem $\dot{\mathbf{H}}_O = \mathbf{r} \times \mathbf{F}$, show that the angular momentum $\mathbf{H}_O$ of the particle is conserved.

(**e**) *(7 Points)* Show that one of Lagrange's equations of motion that you calculated in (**b**) is equivalent to the statement that $\mathbf{H}_O \cdot \mathbf{E}_3$ is conserved.

Question 2

A Bead on a Rotating Wire

25 Points

Consider a bead of mass m which is free to move on a rough rod (cf. Figure 2) which is being spun about the vertical with a constant angular speed Ω_0. With the help of a spherical polar coordinate system, the constraints on the particle can be expressed as

$$\dot{\theta} = \Omega_0, \qquad \phi = \phi_0, \tag{2}$$

where Ω_0 and ϕ_0 are constants. The particle is subject to the influence of a gravitational force $-mg\mathbf{E}_3$.

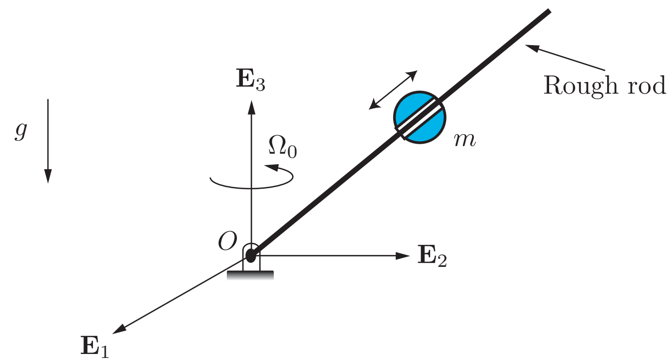


Figure 2: *A particle of mass m which is free to move on a rough rod which is rotating about $\mathbf{E}_3$ with a constant angular speed Ω_0.*

In your answers to the questions below, please make use of the results on spherical polar coordinates on Page 4.

(**a**) *(5 Points)* Suppose that the particle is in motion on the rough rod. What is the velocity vector $\mathbf{v}_{\rm rel}$ of the particle relative to the rod? Give a prescription for the constraint force $\mathbf{F}_c$ acting on the particle.

(**b**) *(13 Points)* Show that the following equations govern the motion of the particle:

$$\begin{aligned} m\ddot{R} - m\Omega_0^2 R \sin^2(\phi_0) + mg\cos(\phi_0) &= \mathbf{F}_c \cdot \tilde{\mathbf{a}}_1, \\ -mR^2 \sin(\phi_0)\cos(\phi_0)\,\Omega_0^2 - mgR\sin(\phi_0) &= \mathbf{F}_c \cdot \tilde{\mathbf{a}}_2, \\ \frac{d}{dt}\left(mR^2\sin^2(\phi_0)\,\Omega_0\right) &= \mathbf{F}_c \cdot \tilde{\mathbf{a}}_3. \end{aligned} \tag{3}$$

In your solution, give clear expressions for $\mathbf{F}_c \cdot \tilde{\mathbf{a}}_k$.

(**c**) *(7 Points)* Suppose that the particle is stationary on the rod: $\mathbf{v}_{\rm rel} = \mathbf{0}$ and $R = R_0$. Give a prescription for the constraint force $\mathbf{F}_c$ in this case. Then, with the help of (3), show that, if

$$R_0 = \left(\frac{g}{\Omega_0^2}\right)\frac{\cos(\phi_0)}{\sin^2(\phi_0)}, \tag{4}$$

then the static friction force acting on the particle is $\mathbf{0}$.

Question 3

A Curvilinear Coordinate System

30 Points

Consider the following curvilinear coordinate system for $\mathbb{E}^3$:

$$q^1 = r = \sqrt{x^2 + y^2}, \qquad q^2 = \theta = \arctan\left(\frac{y}{x}\right), \qquad q^3 = z - \beta r^2, \tag{5}$$

where $\beta > 0$ is a constant. An example of the graph of $z = \beta r^2 + c_0$ is shown in Figure 3.

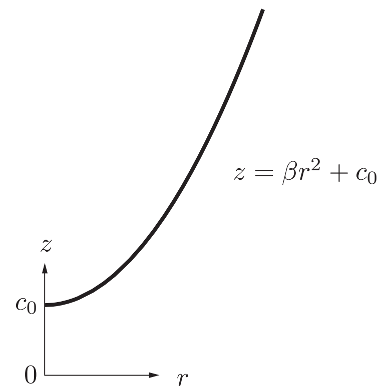


Figure 3: *Graph of the function $z = \beta r^2 + c_0$ when c_0 and β are positive constants.*

In your answers to the questions below, please make use of the results on cylindrical polar coordinates on Page 5.

(**a**) *(6 Points)* Show that the covariant basis vectors for the coordinate system (5) are

$$\mathbf{a}_1 = \mathbf{e}_r + 2\beta r \mathbf{E}_3, \qquad \mathbf{a}_2 = r\mathbf{e}_\theta, \qquad \mathbf{a}_3 = \mathbf{E}_3. \tag{6}$$

(**b**) *(6 Points)* Verify that the contravariant basis vectors for the coordinate system (5) are

$$\mathbf{a}^1 = \mathbf{e}_r, \qquad \mathbf{a}^2 = \frac{1}{r}\mathbf{e}_\theta, \qquad \mathbf{a}^3 = -2\beta r \mathbf{e}_r + \mathbf{E}_3. \tag{7}$$

(**c**) *(8 Points)* On a $q^2 = \theta$ coordinate surface, draw representative examples of the covariant basis vectors $\mathbf{a}_1$ and $\mathbf{a}_3$, representative examples of the contravariant basis vectors $\mathbf{a}^1$ and $\mathbf{a}^3$, and representative examples of the q^1 and q^3 coordinate curves. (Points will not be awarded if your figure is illegible).

(**d**) *(10 Points)* Consider a particle of mass m moving on a smooth paraboloid of revolution:

$$z - \beta r^2 = 0, \tag{8}$$

where β is a positive constant. In addition to a constraint force, a vertical gravitational force $-mg\mathbf{E}_3$ acts on the particle. Show that the equations of motion for the particle can be reduced to a single differential equation:

$$m\left(1 + 4\beta^2 r^2\right)\ddot{r} + 4m\beta^2 r\dot{r}^2 - \frac{h^2}{mr^3} = -2mg\beta r, \tag{9}$$

where h is a constant:

$$h = mr^2\dot{\theta}. \tag{10}$$

Notes on Spherical Polar Coordinates

Recall that the spherical polar coordinates $\{R, \phi, \theta\}$ are defined using Cartesian coordinates $\{x = x_1, y = x_2, z = x_3\}$ by the relations:

$$R = \sqrt{x_1^2 + x_2^2 + x_3^2}, \quad \theta = \arctan\left(\frac{x_2}{x_1}\right), \quad \phi = \arctan\left(\frac{\sqrt{x_1^2 + x_2^2}}{x_3}\right).$$

In addition, it is convenient to define the following orthonormal basis vectors:

$$\begin{bmatrix} \mathbf{e}_R \\ \mathbf{e}_\phi \\ \mathbf{e}_\theta \end{bmatrix} = \begin{bmatrix} \cos(\theta)\sin(\phi) & \sin(\theta)\sin(\phi) & \cos(\phi) \\ \cos(\theta)\cos(\phi) & \sin(\theta)\cos(\phi) & -\sin(\phi) \\ -\sin(\theta) & \cos(\theta) & 0 \end{bmatrix} \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{bmatrix}.$$

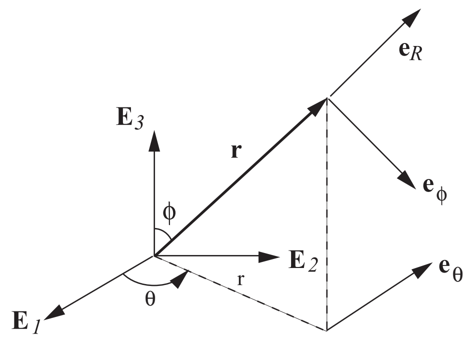


Figure 4: Spherical polar coordinates

For the coordinate system $\{R, \phi, \theta\}$, the covariant basis vectors are

$$\mathbf{a}_1 = \mathbf{e}_R, \quad \mathbf{a}_2 = R\mathbf{e}_\phi, \quad \mathbf{a}_3 = R\sin(\phi)\mathbf{e}_\theta.$$

In addition, the contravariant basis vectors are

$$\mathbf{a}^1 = \mathbf{e}_R, \quad \mathbf{a}^2 = \frac{1}{R}\mathbf{e}_\phi, \quad \mathbf{a}^3 = \frac{1}{R\sin(\phi)}\mathbf{e}_\theta.$$

For a particle of mass m which is unconstrained, the linear momentum $\mathbf{G}$, angular momentum $\mathbf{H}_O$ and kinetic energy T of the particle are

$$\begin{aligned} \mathbf{G} &= m\dot{R}\mathbf{a}_1 + m\dot{\phi}\mathbf{a}_2 + m\dot{\theta}\mathbf{a}_3, \\ \mathbf{H}_O &= mR^2\left(\dot{\phi}\mathbf{e}_\theta - \dot{\theta}\sin(\phi)\mathbf{e}_\phi\right), \\ T &= \frac{m}{2}\left(\dot{R}^2 + R^2\dot{\phi}^2 + R^2\sin^2(\phi)\dot{\theta}^2\right). \end{aligned}$$

Notes on Cylindrical Polar Coordinates

Recall that the cylindrical polar coordinates $\{r, \theta, z\}$ are defined using Cartesian coordinates $\{x = x_1, y = x_2, z = x_3\}$ by the relations:

$$r = \sqrt{x_1^2 + x_2^2}, \quad \theta = \arctan\left(\frac{x_2}{x_1}\right), \quad z = x_3.$$

In addition, it is convenient to define the following orthonormal basis vectors:

$$\begin{bmatrix} \mathbf{e}_r \\ \mathbf{e}_\theta \\ \mathbf{e}_z \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{bmatrix}.$$

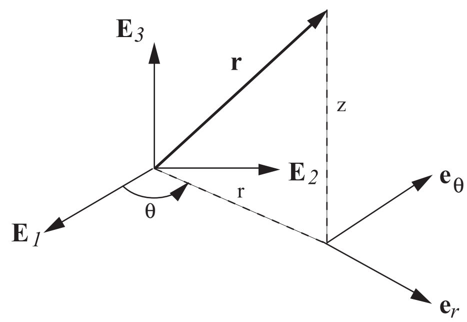


Figure 5: Cylindrical polar coordinates

For the coordinate system $\{r, \theta, z\}$, the covariant basis vectors are

$$\mathbf{a}_1 = \mathbf{e}_r, \quad \mathbf{a}_2 = r\mathbf{e}_\theta, \quad \mathbf{a}_3 = \mathbf{e}_z.$$

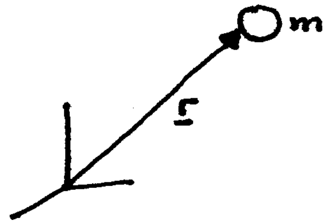
In addition, the contravariant basis vectors are

$$\mathbf{a}^1 = \mathbf{e}_r, \quad \mathbf{a}^2 = \frac{1}{r}\mathbf{e}_\theta, \quad \mathbf{a}^3 = \mathbf{e}_z.$$

The gradient of a function $u(r, \theta, z)$ has the representation

$$\nabla u = \frac{\partial u}{\partial r}\mathbf{e}_r + \frac{\partial u}{\partial \theta}\frac{1}{r}\mathbf{e}_\theta + \frac{\partial u}{\partial z}\mathbf{E}_3.$$

Problem 1



Particle subject to a force $\underline{F}_n$

$$\underline{F}_n = -\frac{Gmm}{R^2}\underline{e}_R$$

The potential energy associated with this force is $U = -\dfrac{Gmm}{R}$

(a) $U = -\dfrac{GMm}{R}$ $\qquad T = \frac{1}{2}m(\dot{R}^2 + R^2\dot{\theta}^2\sin^2\phi + R^2\dot{\phi}^2)$

$$L = T - U$$

(b)
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{R}} = m\dot{R}\right) - \left(\frac{\partial L}{\partial R} = m(R\dot{\phi}^2 + R\dot{\theta}^2\sin^2\phi) - \frac{Gmm}{R^2}\right) = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}} = mR^2\sin^2\phi\,\dot{\theta}\right) - \left(\frac{\partial L}{\partial \theta} = 0\right) = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}} = mR^2\dot{\phi}\right) - \left(\frac{\partial L}{\partial \phi} = mR^2\dot{\theta}^2\sin\phi\cos\phi\right) = 0$$

(c)
$$\dot{T} = \underline{F}\cdot\underline{v} = -\frac{Gmm}{R^2}\underline{e}_R\cdot\underline{v} = -\frac{Gmm}{R^2}\dot{R} = \frac{d}{dt}\left(\frac{Gmm}{R}\right)$$

But $\dfrac{Gmm}{R} = -U \quad \Rightarrow \quad \dot{T} = -\dot{U}$

$$\Rightarrow \quad \frac{d}{dt}(E = T + U) = 0$$

$\Rightarrow$ E is conserved

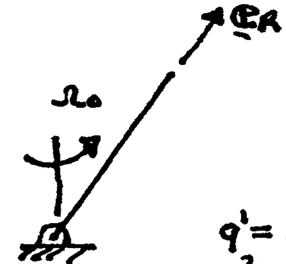
(d)
$$\dot{\underline{H}}_o = \underline{r}\times\underline{F} = \underline{r}\times\left(-\frac{Gm}{\|\underline{r}\|^3}\underline{r}\right) = \underline{0} \quad \text{as } \underline{r}\times\underline{r} = \underline{0}$$

$\Rightarrow$ $\underline{H}_o$ is conserved.

(e)
$$\underline{H}_o\cdot\underline{E}_3 = mR^2(\dot{\phi}\underline{e}_\theta - \dot{\theta}\sin\phi\,\underline{e}_\phi)\cdot\underline{E}_3 = -mR^2\dot{\theta}\sin\phi(-\sin\phi)$$
$$= mR^2\sin^2\phi\,\dot{\theta} = \frac{\partial L}{\partial\dot{\theta}}$$

However from Lagranges equations of motion $\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial\dot{\theta}}\right) = 0 \Rightarrow \dfrac{\partial L}{\partial\dot{\theta}}$ is constant and hence so is $\underline{H}_o\cdot\underline{E}_3$

QUESTION 2



$q^1 = R$, $q^2 = \phi$, $q^3 = \theta$

(a) $$\mathbf{v}_{rel} = \dot{R}\,\mathbf{e}_R$$

$$\mathbf{F}_c = \lambda_1 \mathbf{a}^3 + \lambda_2 \mathbf{a}^2 - \mu_K \|\lambda_1 \mathbf{a}^3 + \lambda_2 \mathbf{a}^2\| \frac{\mathbf{v}_{rel}}{\|\mathbf{v}_{rel}\|}$$

Here $$\mathbf{N} = \lambda_1 \mathbf{a}^3 + \lambda_2 \mathbf{a}^2 \qquad \mathbf{a}^2 = \frac{1}{R}\mathbf{e}_\phi$$

$$\mathbf{F}_c = \mathbf{N} + \mathbf{F}_f \quad , \quad \|\mathbf{v}_{rel}\| = |\dot{R}| \qquad \mathbf{a}^3 = \frac{1}{R\sin\phi}\mathbf{e}_\theta$$

(b) Use Approach I

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}^i}\right) - \frac{\partial T}{\partial q^i} = \mathbf{F}_c \cdot \mathbf{a}_i - mg\,\mathbf{E}_3 \cdot \mathbf{a}_i$$

after evaluating derivatives and substituting constraints $\dot{\theta} = \Omega_0$, $\phi = \phi_0$

$$\frac{d}{dt}(m\dot{R}) - mR\dot{\theta}^2\sin^2\phi - mR\dot{\phi}^2 = \mathbf{F}\cdot\mathbf{e}_R$$

$$\Rightarrow \quad m\ddot{R} - mR\,\Omega_0^2\sin^2\phi_0 = -mg\cos\phi + \mathbf{F}_f\cdot\mathbf{e}_R$$

$$\frac{d}{dt}\left(mR^2\dot{\phi}\right) - mR^2\dot{\theta}^2\sin\phi\cos\phi = \mathbf{F}\cdot R\,\mathbf{e}_\phi$$

$$\Rightarrow \quad - mR^2\Omega_0^2\sin\phi_0\cos\phi_0 = Rmg\sin\phi + \lambda_2$$

$$\frac{d}{dt}\left(mR^2\sin^2\phi\,\dot{\theta}\right) = \mathbf{F}\cdot R\sin\phi_0\,\mathbf{e}_\theta$$

$$\Rightarrow \quad \frac{d}{dt}\left(mR^2\sin^2\phi_0\,\Omega_0\right) = \lambda_1$$

In these equations $$\mathbf{F}_f\cdot\mathbf{e}_R = \|\lambda_1\mathbf{a}^3 + \lambda_2\mathbf{a}^2\|\frac{\dot{R}}{|\dot{R}|}$$

Note that you can't use Approach II here. If you did then you would only get 1 equation — the $m\ddot{R}$ one and be unable to compute λ_1 and λ_2

(c) If particle is stationary

$$\underline{F_c} = \lambda_1 \underline{a}^3 + \lambda_2 \underline{a}^2 + \lambda_3 \underline{a}^1$$

which is equivalent to $\underline{F_c} = \underline{N} + \underline{F_f}$

where $\underline{N} = \lambda_1 \underline{a}^3 + \lambda_2 \underline{a}^2$; $\underline{F_f} = \lambda_3 \underline{a}^1$

Lagranges equations in this case can be inferred from (b) by replacing $\underline{F_f}$ with a static friction force and setting $\dot{R} = 0$

$$\Rightarrow \qquad \underline{F_f} \cdot \underline{e}_R = mg \cos\phi - m R_0 \Omega_0^2 \sin^2\phi_0$$

Now $\underline{F_f}$ only has an $\underline{e}_R$ component so $\underline{F_f} = \underline{0} \Leftrightarrow \underline{F_f} \cdot \underline{e}_R = 0$

Hence

$$mg \cos\phi = m R_0 \Omega_0^2 \sin^2\phi_0$$

$$\Rightarrow \qquad R_0 = \frac{g}{\Omega_0^2} \frac{\cos\phi_0}{\sin^2\phi_0}$$

and when R_0 has this value $\underline{F_f} = \underline{0}$.

QUESTION 3

$q^1 = r \qquad q^2 = \theta$

$q^3 = z - \beta r^2$

a) $\underline{r} = r\,\underline{e}_r + z\,\underline{E}_3$

$\quad = q^1\,\underline{e}_r + (q^3 + \beta q^1 q^1)\,\underline{E}_3$

Hence $\quad \underline{a}_1 = \dfrac{\partial \underline{r}}{\partial r} = \underline{e}_r + 2\beta r\,\underline{E}_3$

$\underline{a}_2 = \dfrac{\partial \underline{r}}{\partial \theta} = r\,\underline{e}_\theta$

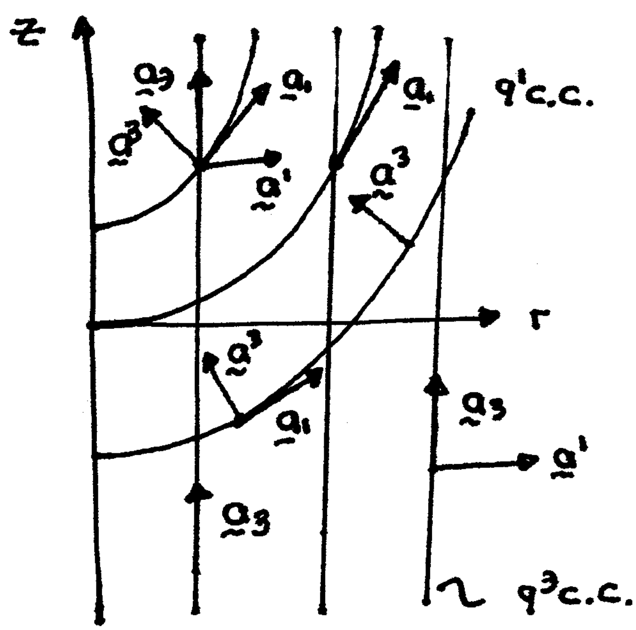
$\underline{a}_3 = \dfrac{\partial \underline{r}}{\partial q^3} = \underline{E}_3$

(b) Easiest to use $\nabla u = \dfrac{\partial u}{\partial r}\underline{e}_r + \dfrac{1}{r}\dfrac{\partial u}{\partial \theta}\underline{e}_\theta + \dfrac{\partial u}{\partial z}\underline{E}_3$. Then we compute that

$\underline{a}^1 = \nabla r = \underline{e}_r \qquad \underline{a}^2 = \nabla\theta = \frac{1}{r}\underline{e}_\theta \quad , \quad \underline{a}^3 = \nabla q^3 = \underline{E}_3 - 2\beta r\,\underline{e}_r$

It's easy to check that $\underline{a}^i \cdot \underline{a}_K = \delta^i_K$.

(c)



$\underline{a}^1 = \underline{e}_r$

$\underline{a}_3 = \underline{E}_3$

A common error here was to let $r < 0$

(d) Particle moving on a q^3 c. surface

$$q^3 = z - \beta r^2 = 0$$

Surface is smooth so $\underline{F_c} = \lambda \underline{a}^3$

Use Approach II

$$\tilde{T} = \tfrac{1}{2} m \underline{\tilde{v}} \cdot \underline{\tilde{v}} \qquad \text{where} \qquad \underline{\tilde{v}} = \dot{r}(\underline{e}_r + 2\beta r \underline{E}_3) + r\dot{\theta}\underline{e}_\theta$$

Hence $$\tilde{T} = \tfrac{1}{2} m \left(\dot{r}^2 (1 + 4\beta^2 r^2) + r^2 \dot{\theta}^2 \right)$$

Now

$$\frac{d}{dt}\left[\frac{\partial \tilde{T}}{\partial \dot{r}} = m(1+4\beta^2 r^2)\dot{r}\right) - \left(\frac{\partial T}{\partial r} = 4m\beta^2 r \dot{r}^2 + mr\dot{\theta}^2\right)$$

$$= \underline{F} \cdot \underline{\tilde{a}}_1 = -2mg\beta r$$

$$\frac{d}{dt}\left(\frac{\partial \tilde{T}}{\partial \dot{\theta}} = mr^2\dot{\theta}\right) - \left(\frac{\partial \tilde{T}}{\partial \theta} = 0\right) = \underline{F} \cdot \underline{\tilde{a}}_2 = \underline{F} \cdot r\underline{e}_\theta = 0$$

Hence (after cancelling $-4m\beta^2 r \dot{r}^2$ with $8m\beta^2 r \dot{r}^2$ to get $+4m\beta^2 \dot{r}^2 r$)

$$m\ddot{r}(1 + 4\beta^2 r^2) + 4m\beta^2 r \dot{r}^2 - mr\dot{\theta}^2 = -2mg\beta r$$

$$\frac{d}{dt}(mr^2\dot{\theta}) = 0$$

Now using angular momentum conservation,

$$h = mr^2\dot{\theta}, \qquad \dot{h} = 0,$$

we find that

$$m(1 + 4\beta^2 r^2)\ddot{r} + 4m\beta^2 r \dot{r}^2 - \frac{h^2}{mr^3} = -2mg\beta r$$

which was to be shown.