Midterm 1 Solution

CBE 162, Spring 2017

March 21, 2017

Problem 1 (60 pts). A jacketed, well-mixed vessel for heating a liquid medium using condensing steam is shown in Figure 1. The vessel is cylindrical with radius r. The inlet temperature and volumetric flow of the liquid are T_f and q_f, respectively. The process inputs include volumetric flow rates q_f and q as well as the steam temperature T_s.The liquid temperature T in the tank and the tank level h are the measured process outputs.

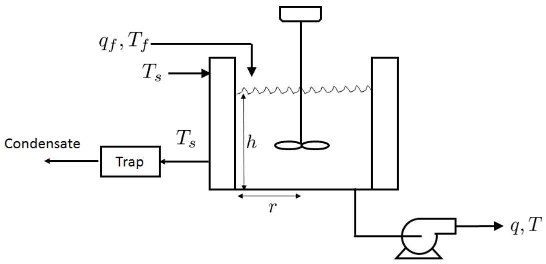

Figure 1: A jacketed, well-mixed heating vessel.

a. (20 pts) Derive a linear dynamic model for the process at hand. State all your modeling assumptions.

Assuming (3 pts) for at least 3 assumptions

- Shaft work is negligible
- Kinetic and potential energy terms in energy balance is negligible
- Liquid is incompressible i.e., constant density
- Constant c_p
- Steam in jacket does not appreciable cool i.e, is at constant temperature T_s.
- Loss of mass due to evaporation is negligible

Overall mass balance (3 pts)

$$\rho \frac{dV}{dt} = \rho A_c \frac{dh}{dt} = \rho q_f - \rho q \tag{1}$$

$$A_c \frac{dh}{dt} = q_f - q \tag{2}$$

Energy balance (6 pts)

$$\rho c_p \frac{dVT}{dt} = \rho c_p q_f (T_f - T_{ref}) - \rho c_p q (T - T_{ref}) + UA_h(T_s - T) \tag{3}$$

$$T\frac{dV}{dt} + V\frac{dT}{dt} = q_f(T_f - T_{ref}) - q(T - T_{ref}) + \frac{UA_h}{\rho c_p}(T_s - T) \tag{4}$$

Substituting (2)

$$hA_c\frac{dT}{dt} = q_f(T_f - T_{ref}) - q(T - T_{ref}) - T(q_f - q) + \frac{UA_h}{\rho c_p}(T_s - T) \tag{5}$$

$$\frac{dT}{dt} = \frac{q_f}{hA_c}(T_f - T - T_{ref}) + \frac{q}{hA_c}(T_{ref}) + \frac{UA_h}{\rho c_p hA_c}(T_s - T) \tag{6}$$

let T_{ref}=0 and with tank perimeter p_T, heat transfer area A_h=$p_T h$,

$$\frac{dT}{dt} = \frac{q_f}{hA_c}(T_f - T) + \frac{Up_T}{\rho c_p A_c}(T_s - T) \tag{7}$$

(2) is already linear. In terms of deviation variables:

$$A_c\frac{d\bar{h}}{dt} = \bar{q}_f - \bar{q} \tag{8}$$

Linearizing (5), (8 pts)

$$\frac{d\bar{T}}{dt} \sim \frac{(T_{f,ss} - T_{ss})}{A_c h_{ss}}\bar{q}_f + \frac{Up_T}{\rho c_p A_c}\bar{T}_s \tag{9}$$

$$- \left(\frac{q_{f,ss}}{A_c h_{ss}} + \frac{Up_T}{\rho c_p A_c}\right)\bar{T} - \left(\frac{q_{f,ss}(T_{f,ss} - T_{ss})}{A_c h_{ss}^2}\right)\bar{h} \tag{10}$$

b. (12 pts) Define state, input, and output vectors. Express the model equations obtained in part a in the linear state-space form.

The linear state-space model takes the form

$$\frac{dx}{dt} = Ax + Bu \tag{11}$$

$$y = Cx + Du \tag{12}$$

With state and input and output vectors (1 pt)

$$x = \begin{bmatrix} \bar{T} \\ \bar{h} \end{bmatrix} \qquad u = \begin{bmatrix} \bar{q}_f \\ \bar{q} \\ \bar{T}_s \end{bmatrix} \qquad y = x = \begin{bmatrix} \bar{T} \\ \bar{h} \end{bmatrix} \tag{13}$$

The model equations are (10 pts) 1 for each matrix element

$$\frac{dx}{dt} = \frac{d}{dt}\begin{bmatrix} \bar{T} \\ \bar{h} \end{bmatrix} = \begin{bmatrix} -\left(\frac{q_{f,ss}}{A_c h_{ss}} + \frac{U p_T}{\rho c_p A_c}\right) & -\left(\frac{q_{f,ss}(T_{f,ss}-T_{ss})}{A_c h_{ss}^2}\right) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{T} \\ \bar{h} \end{bmatrix} \tag{14}$$

$$+ \begin{bmatrix} \frac{(T_{f,ss}-T_{ss})}{A_c h_{ss}} & 0 & \frac{U p_T}{\rho c_p A_c} \\ \frac{1}{A_c} & -\frac{1}{A_c} & 0 \end{bmatrix} \begin{bmatrix} \bar{q}_f \\ \bar{q} \\ \bar{T}_s \end{bmatrix} \tag{15}$$

and (1 pts) for correct C matrix

$$y = \begin{bmatrix} \bar{T} \\ \bar{h} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{T} \\ \bar{h} \end{bmatrix} \tag{16}$$

c. (20 pts) Derive the transfer functions relating the tank level h and the liquid temperature T to the three process inputs. Clearly define any lumped parameters you use.

Let (2 pts) for lumped parameter definitions

$$\alpha_1 = \frac{q_{f,ss}}{A_c h_{ss}} + \frac{U p_T}{\rho c_p A_c} \quad \alpha_2 = \frac{q_{f,ss}(T_{f,ss} - T_{ss})}{A_c h_{ss}^2} \quad \alpha_3 = \frac{(T_{f,ss} - T_{ss})}{A_c h_{ss}} \tag{17}$$

$$\alpha_4 = \frac{U p_T}{\rho c_p A_c} \qquad \alpha_5 = A_c \tag{18}$$

re-writing the linearized model equations

$$\alpha_5 \frac{d\bar{h}}{dt} = \bar{q}_f - \bar{q}$$
$$\frac{d\bar{T}}{dt} = \alpha_3 \bar{q}_f + \alpha_4 \bar{T}_s - \alpha_1 \bar{T} - \alpha_2 \bar{h}$$

Taking the Laplace transform of each equation (4 pts) 2 for each

$$sH(s) = \frac{Q_f(s)}{\alpha_6} - \frac{Q(s)}{\alpha_6} \tag{19}$$

$$sT(s) = \alpha_3 Q_f(s) + \alpha_4 T_s(s) - \alpha_1 T(s) - \alpha_2 H(s) \tag{20}$$

Rearrange to obtain the transfer functions between h and q_f and q. Fluid height is not related to the steam temperature T_s. (6 pts) 2 for each transfer function

$$H(s) = \frac{1}{\alpha_6 s} Q_f(s) - \frac{1}{\alpha_6 s} Q(s) + 0 T_s(s)$$

For the temperature transfer functions, substitute for H(s) (2 pts) for correct substitution

$$(s + \alpha_1) T(s) = \alpha_3 Q_f(s) + \alpha_4 T_s(s) - \frac{\alpha_2}{\alpha_6 s} Q_f(s) + \frac{\alpha_2}{\alpha_6 s} Q(s) \tag{21}$$

Re-arrange to obtain, (6 pts) 2 for each transfer function

$$T(s) = \frac{\alpha_4}{s + \alpha_1} T_s(s) + \left(\alpha_3 - \frac{\alpha_2}{\alpha_6 s} \right) \frac{1}{s + \alpha_1} Q_f(s) + \left(\frac{\alpha_2}{\alpha_6 s} \right) \frac{1}{s + \alpha_1} Q(s) \tag{22}$$

$$T(s) = \frac{\alpha_4}{s + \alpha_1} T_s(s) + \frac{\alpha_3 \alpha_6 s - \alpha_2}{\alpha_6 s (s + \alpha_1)} Q_f(s) + \frac{\alpha_2}{\alpha_6 s (s + \alpha_1)} Q(s) \tag{23}$$

d. (8pts) Can the response of the liquid temperature T to any of the process inputs show a nonminimum phase behavior? If so, derive the condition(s) under which the nonminimum phase behavior is observed and provide a physical interpretation for this behavior.

The transfer function between $\frac{T(s)}{Q_f(s)}$ exhibits numerator dynamics. Nonminimum phase behavior relates to positive zeros (2pts), hence the conditions for inverse response are (2 pts)

$$\frac{\alpha_2}{\alpha_3 \alpha_6} > 0 \tag{24}$$

Note α_6 must be positive by definition. Then the condition reduces to (2pts)

$$\frac{\alpha_2}{\alpha_3} > 0 \tag{25}$$

$$\frac{\alpha_2}{\alpha_3} = \frac{1}{h_{ss}} > 0, \tag{26}$$

Since tank height can't be negative, the above condition holds and inverse response will be observed. (2pts)

Problem 2. (50 pts) We aim to design a closed-loop control system for a gas absorber that reduces the concentration of SO_2 in a gas effluent via manipulating the inlet water flow rate to the absorber. The first-order dynamics of the gas absorber can be described by

$$G_p(s) = \frac{\bar{C}(s)}{\bar{F}(s)} = \frac{-0.05}{2s+1},$$

where $\bar{C}(t)$ is the effluent SO_2 concentration and $\bar{F}(t)$ is the water flow rate, both defined in terms of deviation variables based on the steady-state values $C_{ss} = 100$ ppm and $F_{ss} = 250$ gpm. The characteristic equation of the closed-loop system is given by $1 + G_c G_v G_p G_m = 0$, where G_c, G_v, and G_m denote the transfer functions for the controller, valve, and measurement sensor, respectively.

a. (3 pts) We first intend to design a proportional-only controller with positive gain for this system. Assuming that the sensor has a positive gain, determine whether the gain of the valve can be positive. Explain your choice.

If the gain of the valve is positive, then a positive error $Y_{sp} - Y$ increases the water flow rate. This means that when the setpoint is higher than the current SO_2 concentration, decreases the concentration decreases and moves farther from the setpoint. Therefore, the valve gain cannot be positive.
(+2) for correct answer
(+1) for explanation

b. (14 pts) Assume $G_m = 1$ mV/ppm and $G_v = -1$ gpm/mV (note that the voltage output of the concentration analyzer is in mV). Consider a PI controller with the integral time constant $\tau_I = 0.5$ min. Determine the range of all possible values for the proportional gain K_c. The transfer function for a PI controller is given by

$$G_c(s) = K_c \left(1 + \frac{1}{\tau_I s}\right).$$

For the closed loop to be stable, the roots of the characteristic equation must be less than zero.

$$1 + G_c G_v G_p G_m = 0$$

$$1 + K_c \left(1 + \frac{1}{\tau_I s}\right)(-1)\frac{-0.05}{2s+1}(1) = 0$$

$$1 + \frac{0.05K_c(0.5s+1)}{0.5s(2s+1)} = 0$$

$$0.5s(2s+1) + 0.05K_c(0.5s+1) = 0$$

$$s^2 + (0.5 + 0.025K_c)s + 0.05K_c = 0$$

$$s = \frac{-(0.5+0.025K_c) \pm \sqrt{(0.5+0.025K_c)^2 - 4(0.05K_c)}}{2} < 0$$

Root 1:

$$-(0.5 + 0.025K_c) + \sqrt{(0.5 + 0.025K_c)^2 - 4(0.05K_c)} < 0$$

$$\sqrt{(0.5 + 0.025K_c)^2 - 4(0.05K_c)} < (0.5 + 0.025K_c)$$

$$(0.5 + 0.025K_c)^2 - 4(0.05K_c) < (0.5 + 0.025K_c)^2$$

$$K_c < 0$$

Root 2:

$$-(0.5 + 0.025K_c) - \sqrt{(0.5 + 0.025K_c)^2 - 4(0.05K_c)} < 0$$

$$K_c > 0$$

(+2) for stability criterion
(+4) for writing out closed-loop characteristic equation
(+2) for substituting values in closed-loop characteristic equation
(+2) for quadratic formula
(+4) for correct range of K_c

c. (13 pts) Using the PI controller of part b, we would like to achieve a closed-loop response that is underdamped with a damping ratio of 0.5. Determine the proportional gain K_c of the PI controller such that the desired closed-loop response can be achieved. *Note: You may leave your answer in the quadratic formula, if necessary.*

From part b,

$$s^2 + (0.5 + 0.025K_c)s + 0.05K_c = 0$$

$$\frac{1}{0.05K_c}s^2 + \frac{(0.5 + 0.025K_c)}{0.05K_c}s + 1 = 0$$

$$\tau^2 = \frac{1}{0.05K_c}$$

$$\tau = \sqrt{\frac{1}{0.05K_c}}$$

$$2\tau\zeta = \frac{0.5 + 0.025K_c}{0.05K_c}$$

$$2(0.5)\sqrt{\frac{1}{0.05K_c}} = \frac{0.5 + 0.025K_c}{0.05K_c}$$

$$\sqrt{0.05K_c} = 0.5 + 0.025K_c$$

$$0.05K_c = 0.25 + 0.025K_c + 6.25 \times 10^{-4}K_c^2$$

$$6.25 \times 10^{-4}K_c^2 - 0.025K_c + 0.25 = 0$$

$$K_c = \frac{0.025 \pm \sqrt{(-0.025)^2 - 4(6.25 \times 10^{-4})(0.25)}}{2(6.25 \times 10^{-4})}$$

$$K_c = 20$$

(+2) for writing down characteristic equation in standard form
(+2) for solving for τ
(+2) for solving for $2\tau\zeta$
(+2) for substituting in values for τ and ζ
(+2) for quadratic formula
(+3) for final answer

d. (20 pts) Now reality strikes! We can no longer ignore the sensor and valve dynamics. The voltage output of the gas analyzer is calibrated as

$$V(t) = 0.45 + 1.2C(t),$$

where the voltage output $V(t)$ is in mV and the SO_2 concentration $C(t)$ is in ppm. In the calibration model above, 0.45 corresponds to the base analyzer output at steady state. The gas analyzer has a sampling line with a time delay of 0.2 min. The valve exhibits first-order dynamics with a gain of − 0.9 gpm/mV and a time constant of 0.3 min.

EPA is imposing stringent regulations on the output SO_2 concentration of the absorber. For the controller

$$G_c(s) = 25\left(1 + \frac{1}{0.5s}\right)$$

obtain the transfer function between the concentration setpoint and the measured concentration of the absorber. Analyze the closed-loop response in terms of its initial and steady-state behavior when the setpoint concentration is abruptly changed from its steady-state value of 100 ppm to 90 ppm.

Take the Laplace transform:

$$G_m = \frac{\bar{V}(s)}{\bar{C}(s)} = 1.2e^{-0.2s}$$

$$G_v = \frac{-0.9}{0.3s+1}$$

Closed Loop response:

$$1 + G_cG_vG_pG_m = 0$$

$$1 + 25\left(\frac{0.5s+1}{0.5s}\right)\left(\frac{-0.9}{0.3s+1}\right)\left(\frac{-0.05}{2s+1}\right)(1.2e^{-0.2s}) = 0$$

Use a first-order Pade approximation:

$$1 + 25\left(\frac{0.5s+1}{0.5s}\right)\left(\frac{-0.9}{0.3s+1}\right)\left(\frac{-0.05}{2s+1}\right)\left(\frac{1.2(-0.1s+1)}{0.1s+1}\right) = 0$$

Transfer Function:

$$\frac{Y}{Y_{sp}} = \frac{G_cG_vG_p}{1 + G_cG_vG_pG_m}$$

$$\frac{Y}{Y_{sp}} = \frac{25\left(\frac{0.5s+1}{0.5s}\right)\left(\frac{-0.9}{0.3s+1}\right)\left(\frac{-0.05}{2s+1}\right)}{1 + 25\left(\frac{0.5s+1}{0.5s}\right)\left(\frac{-0.9}{0.3s+1}\right)\left(\frac{-0.05}{2s+1}\right)\left(\frac{1.2(-0.1s+1)}{0.1s+1}\right)}$$

Final Value Theorem

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s)$$

$$\lim_{s\to 0} sY(s) = s \cdot \frac{25\left(\frac{0.5s+1}{0.5s}\right)\left(\frac{-0.9}{0.3s+1}\right)\left(\frac{-0.05}{2s+1}\right)}{1 + 25\left(\frac{0.5s+1}{0.5s}\right)\left(\frac{-0.9}{0.3s+1}\right)\left(\frac{-0.05}{2s+1}\right)\left(\frac{1.2(-0.1s+1)}{0.1s+1}\right)} \cdot \frac{-10}{s}$$

$$\lim_{s\to 0} sY(s) = \frac{25(0.5s+1)(-0.9)(-0.05)(0.1s+1)(-10)}{(0.5s)(0.3s+1)(2s+1)(0.1s+1)+25(0.5s+1)(-0.9)(-0.05)(1.2)(-0.1s+1)}$$

$$\lim_{s\to 0} sY(s) = \frac{25(1)(-0.9)(-0.05)(1)(-10)}{25(1)(-0.05)(-0.9)(1.2)(1)}$$

$$\lim_{s\to 0} sY(s) = \frac{-10}{1.2} = -8.33$$

(+2) for transfer function G_m
(+2) for transfer function G_v
(+2) for plugging in time delay
(+4) for writing transfer function Y/Y_{sp}
(+3) for plugging in expressions for G_c,G_v,G_p,G_m
(+3) for writing final value theorem
(+2) for correct input $U(s)$
(+2) for final answer