MT2 Solutions

1. (a) For a potential with two sloped walls, we have the following quantization condition (see Griffiths pg 332).

$$\int_{x_-}^{x_+} \sqrt{2m(E - V(x))}dx = \pi\hbar(n - 1/2)$$

where $x_\pm = \pm\sqrt{\frac{2E}{m\omega^2}}$, are the classical turning points.
By symmetry:

$$2\int_0^{x_+} \sqrt{2m\frac{m\omega^2 x_+^2}{2}(1 - x^2/x_+^2)}dx = \pi\hbar(n - 1/2)$$

Using the proved integral with $b = 1/x_2$:

$$2m\omega x_+ \left[\frac{1}{2}x\sqrt{1 - x^2/x_+^2} + \frac{x_2 \arcsin(x/x_+)}{2}\right]_0^{x_2} = \pi\hbar(n - 1/2)$$

$$\frac{\pi m\omega x_+^2}{2} = \pi\hbar(n - 1/2)$$

If we substitute $x_+ = \sqrt{\frac{2E}{m\omega^2}}$, we find:

$$E = \hbar\omega(n - 1/2)$$

We see WKB gives the exact energies for a SHO is this case. (Don't let the $-1/2$ bother you, its just a statement that we begin indexing our solutions at $n = 1$ instead of $n = 0$.

(b) At energies where the turning point is $x = |a|$, the quantization condition becomes:

$$2\int_0^{a} \sqrt{2m\frac{m\omega x_0^2}{2}(1 - x^2/x_0^2)}dx = n\hbar\pi$$

where $x_0 = \sqrt{2E/(m\omega^2)}$. The integral has exactly the same form as that in part a (but with different bounds). When evaluated, we find:

$$2m\omega x_0 \left[\frac{1}{2}x\sqrt{1 - x^2/x_0^2} + \frac{x_0 \arcsin(x/x_0)}{2}\right]_0^{a} = \pi\hbar n$$

Plugging in, we find:

$$2m\omega x_0\left[\frac{1}{2}a\sqrt{1-a^2/x_0^2}+\frac{x_0\arcsin\left(a/x_0\right)}{2}-\frac{1}{2}a\right]=\pi\hbar n$$

We can simplify ur result if we take $x_0 >> a$, (note there is a typo in the problem). Note to first order, $\arcsin(\epsilon) \approx \epsilon$ and $\sqrt{1-\epsilon} = 1$. Using these results, we find:

$$2m\omega x_0 a = \pi\hbar n \quad \rightarrow \quad E = \frac{(n\pi\hbar)^2}{2ma^2}$$

Is this surprising? Not really, assuming $x_0 >> a$ is equivalent to assuming $E >> \frac{m\omega^2 a^2}{2} \geq V(x)$. In this approximation, $p(x) = \sqrt{2m(E-V(x))} \approx \sqrt{2mE}$. This is exactly the momentum we assign to a particle in confined to in potential well with flat bottom. We have also seen WKB gives the exact particle in a box energy levels for a particle confined to well of length a. (See Griffiths pg 319). This explains the result.

2. Recall:

$$c_b(t) = -\frac{i}{\hbar}\int_0^t \langle b|H'(t')|a\rangle\, e^{i\omega t'}dt'$$

We want to compute $c_b(2\tau+T)$ given $H'(t) = V_{ab}\cos(\omega t)$

$$\begin{aligned}
c_b(2\tau+T) &= -\frac{i}{\hbar}V_{ab}\left[\int_0^{\tau}\cos(\omega t')e^{i\omega_0 t}dt' + \int_{\tau+T}^{2\tau+T}\cos(\omega t')e^{i\omega_0 t'}dt'\right] \\
&= -\frac{1}{2\hbar}V_{ab}\left(\left[\frac{e^{i(\omega_0-\omega)t}}{\omega_0-\omega}\right]_0^{\tau} + \left[\frac{e^{i(\omega_0-\omega)t}}{\omega_0-\omega}\right]_{\tau+T}^{2\tau+T}\right)
\end{aligned}$$

Note: in line 2 I have neglected terms that are suppressed by $1/(\omega_0+\omega)$ as instructed in the problem statement. Letting $\delta = \omega_0 - \omega$ and evaluating the above, we find:

$$\begin{aligned}
c_b &= -\frac{1}{2\hbar}\frac{V_{ab}}{\delta}[e^{i\delta\tau} - 1 + e^{i\delta(T+2\tau)} - e^{i\delta(T+\tau)}] \\
&= -\frac{1}{2\hbar}\frac{V_{ab}}{\delta}(e^{i\delta\tau}-1)(e^{i\delta(T+\tau)}+1) \\
&= -\frac{2i}{\hbar}\frac{V_{ab}}{\delta}\sin(\delta\tau/T)\cos(\delta(\tau+T)/2)e^{i\delta(T+2\tau)/2}
\end{aligned}$$

So we see:

$$P = |c_b|^2 = \frac{4|V_{ab}|^2}{\hbar^2}\frac{\sin^2(\delta\tau/2)\cos^2(\delta(\tau+T)/2)}{\delta^2}$$

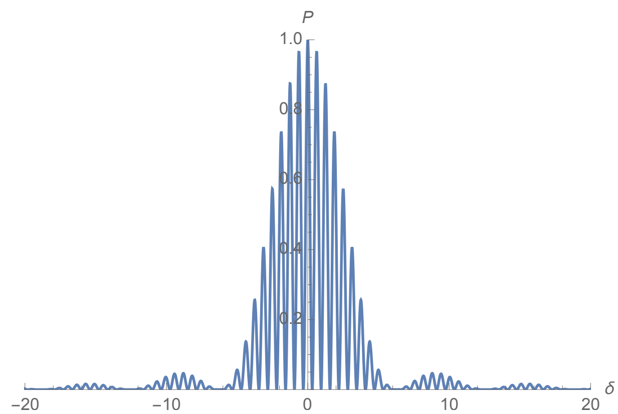
Recalling:

$$\sin(a)\cos(b) = \frac{1}{2}(\sin(a+b)+\sin(a-b))$$

we can write:

$$P = \frac{4|V_{ab}|^2}{\hbar^2} \frac{[\sin(\delta(T+2\tau)/2) - \sin(\delta T/2)]^2}{\delta^2}$$

Below, we plot P vs δ with $T = 9\tau$.



3. (a) We write $V(z)$ as a taylor series (ie. $V(z) \approx V(0) + V'(0)z + \frac{1}{2}V''(0)z^2 + ...$).

$$V'(z) = V_0 k \sin(2kz) \quad \rightarrow \quad V'(0) = 0$$
$$V''(z) = 2V_0 k^2 \cos(2kz) \quad \rightarrow \quad V''(0) = 2V_0 k^2$$

So we find:

$$V \approx V_0 k^2 z^2$$

If we write this as: $\frac{1}{2}m\omega_0^2 z^2$, we identify $\omega_0 = \sqrt{2V_0/m}\ k$

(b) Keeping only terms linear in ϵ, we find the perturbing hamiltonian has the from:

$$H'(t) = m\omega_0^2 \epsilon k z \sin(\omega t)$$

The probability amplitude for finding the particle in state $|1\rangle$ at the end of the perturbation is:

$$\begin{aligned} c_1 &= -\frac{i}{\hbar}\int_0^T m\omega_0^2 \epsilon k \langle 1|z|0\rangle \sin\omega t\ e^{i\omega_{10}t} dt \\ &= -\frac{i}{\hbar} m\omega_0^2 \epsilon k \sqrt{\frac{\hbar}{2m\omega_0}} \int_0^T \sin(\omega t) e^{i\omega_{10}t} dt \\ &= \frac{1}{2\hbar} m\omega_0^2 \epsilon k \sqrt{\frac{\hbar}{2m\omega_0}} \left[\frac{e^{i(\omega_{10}-\omega)t}}{\omega_{10}-\omega}\right]_0^T \end{aligned}$$

Note: Once again we have dropped terms suppressed by $1/(\omega + \omega_0)$. Also note the three different ω in the expression. Here, $\omega_{10} = (E_1 - E_0)/\hbar$. We now finish up the problem.

$$\begin{aligned}
c_1 &= \frac{i}{\hbar} m\omega_0^2 \epsilon k \sqrt{\frac{\hbar}{2m\omega_0}} e^{i(\omega_{10}-\omega)T/2} \frac{\sin(\omega_{10}-\omega)}{\omega_{10}-\omega} \\
&= \frac{\epsilon}{2} m\omega_0^2 \frac{1}{\sqrt{\hbar\omega_0 V_0}} e^{i(\omega_{10}-\omega)T/2} \frac{\sin(\omega_{10}-\omega)}{\omega_{10}-\omega}
\end{aligned}$$

$$P = |c_1|^2 = \frac{\epsilon^2}{4} \frac{m^2\omega_0^3}{\hbar V_0} \frac{\sin^2(\omega_{10}-\omega)}{(\omega_{10}-\omega)^2}$$

4. (a) The Schrodinger equation reads

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V(r)\right)\psi = E\psi$$

Noting that we can write the Laplacian as

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{L^2}{\hbar^2 r^2}$$

taking $\psi = \frac{u(r)}{r} Y_l^m(\theta,\phi)$ gives

$$\begin{aligned}
-\frac{\hbar^2}{2m}&\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\left[\frac{u(r)}{r}\right]\right) Y_l^m(\theta,\phi) - \frac{l(l+1)}{r^2}\frac{u(r)}{r} Y_l^m(\theta,\phi)\right) \\
&= E\frac{u(r)}{r} Y_l^m(\theta,\phi) - V(r)\frac{u(r)}{r} Y_l^m(\theta,\phi)
\end{aligned}$$

using the fact that $L^2 Y_l^m = \hbar^2 l(l+1) Y_l^m$. Since

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\left[\frac{u(r)}{r}\right]\right) = r u''(r)$$

we can divide both sides of (**??**) by $-\frac{\hbar^2 Y_l^m}{2mr^3}$ to yield

$$u''(r) - \frac{2m}{\hbar^2}\left(\frac{\hbar^2 l(l+1)}{2mr^2} + V(r) - E\right) u(r) = 0$$

$$\text{i.e.} \quad u''(r) + \frac{p_{eff}^2(r)}{\hbar^2} u(r) = 0, \quad \text{where}$$

$$\begin{aligned}
p_{eff}(x) &\equiv \sqrt{2m\left(E - V_{eff}(r)\right)} \\
V_{eff}(r) &\equiv \frac{\hbar^2 l(l+1)}{2mr^2} + V(r)
\end{aligned}$$

(b) Taking $u(r) \equiv a(r)e^{if(r)}$ gives

$$u'' = \left(a'' + 2ia'f' - af'^2 + iaf''\right)e^{if}$$

so that (**??**) reads

$$a'' + 2ia'f' + iaf'' - af'^2 + \frac{p_{eff}^2(r)}{\hbar^2}a = 0$$

after dividing through by e^{if}. Now, we assume (without loss of generality) that $a(r)$, $f(r)$ are real, and taking r in the "classical region" such that $p_{eff}(r)$ is also real, we can separate (**??**) into two equations for the real and imaginary parts:

$$\begin{aligned} a'' - af'^2 + \frac{p_{eff}^2(r)}{\hbar^2}a &= 0 \\ 2a'f' + af'' = \left(a^2 f'\right)' &= 0 \end{aligned}$$

The second equation is easily solved:

$$a(r) = \frac{C'}{\sqrt{f'(r)}}$$

for some constant C'. To solve (**??**), assume that $a(r)$ is slowly-varying so that a'' is negligible compared to the other terms. In this case we have

$$\begin{aligned} f'^2 &= \frac{p_{eff}^2(r)}{\hbar^2} \\ \Rightarrow \quad f(r) &= \pm\frac{1}{\hbar}\int p_{eff}(r)dr \end{aligned}$$

so that

$$u(r) \cong \frac{C}{\sqrt{p_{eff}(r)}}e^{\pm\frac{i}{\hbar}\int p_{eff}(r)dr}$$

inputting the solution for $f'(r)$ and absorbing constants into the new constant C. This is the same result as in 1D except that the momentum is replaced by an effective momentum resulting from the effective (centrifugal) potential energy.

(c) In such an "infinite spherical well" potential, our effective 1D potential $V_{eff}(r)$ has one vertical wall at $r = R$ and (for $l > 0$) a sloping wall for $0 < r < R$. Since $u(r)$ has the same form as $\psi(x)$ in the 1D WKB derivation, the 1D connection formulas (Griffiths section 8.3) must also hold. Thus, we can use the usual connection formula for a potential with one vertical wall with the replacement $p(x) \to p_{eff}(r)$ and the Langer correction $l(l+1) \to (l+1/2)^2$, giving

$$\int_{r_1}^{R} p_{eff}(r)dr = \left(n - \frac{1}{4}\right)\pi\hbar$$

$$\Rightarrow \quad \int_{r_1}^{R} \sqrt{2mE - \frac{\hbar^2(l+1/2)^2}{r^2}}dr = \left(n - \frac{1}{4}\right)\pi\hbar$$

where r_1 is the classical turning point such that $V_{eff}(r_1) = E$, so that

$$r_1 = \sqrt{\frac{\hbar^2(l+1/2)^2}{2mE}}$$

5. The total energy is given by:

$$E(\alpha) = \int_{-\infty}^{\infty} \psi^*(x)\frac{-\hbar^2\nabla^2}{2m}\psi(x)dx + \int_{-a/2}^{a/2} \psi^*(x)(-V_0)\psi(x)dx$$

Lets take a look at the kinetic term:

$$\begin{aligned}
\langle T\rangle &= \frac{\hbar^2}{2m}\sqrt{\frac{\alpha}{\pi}}\int_{-\infty}^{\infty} \frac{d\psi^*}{dx}\frac{d\psi}{dx}dx \\
&= \frac{\hbar^2}{2m}\sqrt{\frac{\alpha}{\pi}}\int_{-\infty}^{\infty} \alpha^2 x^2 e^{-\alpha x^2}dx \\
&= \frac{\hbar^2}{2m}\sqrt{\frac{\alpha^5}{\pi}}(-\partial_\alpha)\int_{-\infty}^{\infty} e^{-\alpha x^2}dx \\
&= \frac{\hbar^2}{2m}\sqrt{\frac{\alpha^5}{\pi}}(-\partial_\alpha)\sqrt{\frac{\pi}{\alpha}} \\
&= \frac{\hbar^2\alpha}{4m}
\end{aligned}$$

Now we turn to the potential term:

$$\langle V\rangle = \int_{-a/2}^{a/2} \psi^*(x)(-V_0)\psi(x)dx$$

$$\begin{aligned}
\langle V\rangle &= (-V_0)\int_{-a}^{a} \sqrt{\frac{\alpha}{\pi}}e^{-\alpha x^2}dx \\
&= (-V_0)\int_{-\infty}^{\infty} \sqrt{\frac{\alpha}{\pi}}e^{-\alpha x^2}dx \\
&= -V_0
\end{aligned}$$

In the second line, we have used the fact that $1/\alpha << a^2 \to 1 << \alpha a^2$, to extend the bounds of integration to $\pm\infty$. So we find:

$$E(\alpha) = \frac{\hbar^2 \alpha}{4m} - V_0$$

This is minimized when $\alpha = 0$, and we find $E = -V_0$.