

Solutions to MT2, 2016

May 7, 2016

1 Rabi problem

1.1 Part a

We determine the initial conditions by plugging in $t = 0$ to the given coefficients. This gives us $C_a(0) = 1$, $C_b(0) = 0$.

1.2 Part b

The trick here is to recognize that the new situation (starting in level b) is identical to the old situation (starting in level a) except the energy gap between the level we start in and the other level is now $-\hbar\omega_0$ (instead of positive). Therefore our new solution is simply the given solution, except we switch the labels a and b and switch the sign on δ (this part comes from the energy switching sign).

1.3 Part c

We do this simply by plugging in the given equations (1) and (2), with $\delta = 0$, into (3). This involves taking the time derivative of (1) and (2), then noticing that the time derivative of one just equals the other times the factors given in equation (3).

2 Problem 2

Given a particular perturbation H' to the Hamiltonian, the full Hamiltonian is given by $H = H_0 + H'$. First order non-degenerate perturbation theory tells us that the correction to the ground state ($|0\rangle$) energy is $\langle 0|H'|0\rangle$. We wish to show that this corrected energy always overestimates the energy of the true groundstate $|0'\rangle$. In other words, we wish to show

$$\langle 0|H_0|0\rangle + \langle 0|H'|0\rangle \geq \langle 0'|H|0'\rangle.$$

The LHS reduces to the expectation value of the full Hamiltonian in the old ground state: $\langle 0|H_0|0\rangle + \langle 0|H'|0\rangle = \langle 0|H|0\rangle$. Hence we need to show

$$\langle 0|H|0\rangle \geq \langle 0'|H|0'\rangle.$$

This is manifestly true from the variational principle. In more words: the new ground state $|0'\rangle$ is defined to be the lowest energy eigenstate of the Hamiltonian H . The state $|0\rangle$ can be written as a linear combination of eigenstates of H . The smallest expectation value it can have, then, is equal to the expectation value of H in the state $|0'\rangle$.

3 Problem 3

3.1 Part a

The Hamiltonian for a harmonic oscillator can be written as $H = p^2/2m + \frac{1}{2}k(x - x_0)^2$. Hence we can calculate H' as follows

$$\begin{aligned} H &= p^2/2m + \frac{1}{2}k(x - A \cos \omega t)^2 \\ &= p^2/2m + \frac{1}{2}kx^2 + \frac{1}{2}kA^2 \cos^2 \omega t - xkA \cos \omega t \\ H' &= \frac{1}{2}kA^2 \cos^2 \omega t - xkA \cos \omega t \end{aligned}$$

Griffiths equation [9.28] tells us that sinusoidal perturbations (such as this one) given by perturbations $H' = V(x) \cos \omega t$, the probability of transition is given by

$$P_{1 \rightarrow 2}(t) = \frac{|V_{12}|^2 \sin^2[(\omega_0 - \omega)t/2]}{\hbar^2 (\omega_0 - \omega)^2}$$

where $V_{ab} = \langle 1|V|2\rangle$. In our case, $V_{ab} = \langle 1|-xkA|2\rangle = -kA \langle 1|x|2\rangle = -kA \sqrt{\frac{\hbar}{2m\omega}}$. So there will be a probability of transition! (Notice we discarded the \cos^2 term because it contains no x operators and therefore will not contribute).

3.2 Part b

When we modulate the mass in this problem, we keep k fixed because the problem told us that the spring had a spring constant k . Hence the only change to the energy comes from

the kinetic term:

$$\begin{aligned}
 \frac{p^2}{2m} &= \frac{p^2}{2(m_0 + m' \cos \omega t)} \\
 &= \frac{p^2}{2m_0(1 + \frac{m'}{m_0} \cos \omega t)} \\
 &\approx \frac{p^2}{2m_0} \left(1 - \frac{m'}{m_0} \cos \omega t\right) \\
 H' &= -\frac{p^2}{2m_0^2} m' \cos \omega t
 \end{aligned}$$

This will be unable to transition our system to the first excited state. We see this by writing p in terms of $a^\dagger - a$. No term in p^2 , then, will give a non-zero contribution to V_{12} .

3.3 Part c

First order time-dependent perturbation theory tells us that the amplitude for the first excited state as a function of time is given by the following. Note we have converted from the given force to a potential energy by using the relation $V = -\int F dx = -p_0 \delta(t)x$.

$$\begin{aligned}
 c_2(t) &= -\frac{i}{\hbar} \int_0^t \langle 2 | (-p_0 x \delta(t')) | 1 \rangle e^{i\omega_0 t'} dt' \\
 &= \frac{i}{\hbar} p_0 \sqrt{\frac{\hbar}{2m\omega}} \int_0^t \delta(t') e^{i\omega_0 t'} dt' \\
 &= \frac{i}{\hbar} p_0 \sqrt{\frac{\hbar}{2m\omega}} \\
 P_{12}(t) &= |c_2(t)|^2 = \frac{p_0^2}{2m\omega\hbar}
 \end{aligned}$$

4 Geometric phase

The brute force way to calculate the geometric phase is to plug everything in to the equation for geometric phase as a function of initial and final parameter values, given in Griffiths:

$$\gamma_n(t) = i \int_{R_i}^{R_f} \left\langle \psi_n \left| \frac{\partial \psi_n}{\partial R} \right. \right\rangle dR.$$

However, there is an easier way to argue that the accumulated geometric phase is zero. Simply point out that the wavefunctions are completely real, and the derivative of the wave function will return a real function. Therefore the integral will itself be real-valued. Hence γ will be purely imaginary, but the geometric phase is a real (measurable) quantity. Therefore it must be the case that the geometric phase is zero.