Solutions to MT1, 2016

May 7, 2016

1 Little Calculation Needed

1.1 Part a

These two neutrons are fermions, so their joint wavefunction must be antisymmetric overall. Because they are in the same state of the harmonic oscillator, their position wavefunction is necessarily symmetric. Therefore their spin wavefunction must be antisymmetric. When combining two spin-1/2 particles, there are four possible spin combinations, three of which are symmetric (the $\ell = 1$ spin states), and one of which is antisymmetric (the $\ell = 0$ spin state). Hence their spins must be in the $\ell = 0$ spin state, in which their spins are anti-aligned. The answer to this question is thus that their spins must be oriented oppositely.

1.2 Part b

When a third neutron is added, it must be in the first excited state (because there is no way to fit a third fermion into the ground state). The first excited state has a greater $\langle x^2 \rangle$ than the ground state. Hence, the averaged expectation value increases!

1.3 Part c

The energy of the system goes down! Quick to see by completing the square in the Hamiltonian: $H = p^2/2m + \frac{1}{2}m\omega^2x^2 + \alpha x = p^2/2m + \frac{1}{2}m\omega^2(x^2 - \frac{2\alpha}{m\omega^2}x) = p^2/2m + \frac{1}{2}m\omega^2(x - \frac{\alpha}{m\omega^2})^2 - \alpha^2/2m\omega^2$. We see that the Hamiltonian is exactly the same, except shifted slightly (which doesn't affect the energy) and with an overall negative constant tacked on (lowering the energy).

1.4 Part d

The first order correction to the ground state energy vanishes. This follows directly from expanding x in terms of $a + a^\dagger$, then calculating $\langle 0|a + a^\dagger|0\rangle = 0$.

2 Yukawa potential

2.1 Part a

The energy change to the ground state from the given perturbation is simply

$$\begin{aligned} E^{(1)} &= \langle 1,0,0| A\frac{e^{-r/\lambda}}{r} |1,0,0\rangle \\ &= \frac{4A}{a_0^3}\int e^{-2r/a_0} e^{-r/\lambda} r dr \\ &= \frac{4A\lambda^2}{a_0(a_0+2\lambda)^2} \end{aligned}$$

2.2 Part b

Even though the excited states are degenerate, they have different L^2 and L_z eigenvalues. This allows us to use non-degenerate perturbation theory because those are Hermitian operators that commute with the perturbation Hamiltonian. Said another way, the excited states are the "good" eigenstates with which to do non-degenerate perturbation theory (for this perturbation), which we know because they are different eigenstates of Hermitian operators that commute with the perturbation.

3 Degenerate perturbation theory

3.1 Part a

The strategy here is to find the eigenvectors of the degenerate sub-matrix of H_1 (i.e. the lower right 2x2 sub-matrix). These eigenvectors ψ_{23a} and ψ_{23b} will satisfy $\langle\psi_{23a}| H_1 |\psi_{23b}\rangle = 0$, which means we can use normal degenerate perturbation theory with them. We find the eigenvalues:

$$\begin{aligned} 0 &= (\delta-\lambda)^2 - \gamma^2 \\ \lambda_\pm &= \delta \pm \gamma \end{aligned}$$

The eigenkets are thus (written as a row vector transpose for simplicity, because really they should be column vectors). The T means transpose (turning the row vector into a column vector).

$$\begin{aligned} |\psi_{23a}\rangle &= \frac{1}{\sqrt{2}}(0,i,1)^T \\ |\psi_{23b}\rangle &= \frac{1}{\sqrt{2}}(0,-i,1)^T \end{aligned}$$

3.2 Part b

The first correction to the energy is simply

$$\begin{aligned}
E_{23a}^{(1)} &= \langle\psi_{23a}|H_1|\psi_{23a}\rangle \\
&= \frac{1}{2}(0,-i,1)H_1(0,i,1)^T \\
&= \frac{1}{2}(0,-i,1)(i\Delta - i\sigma, i\delta + i\gamma, \gamma + \delta)^T \\
&= \gamma + \delta
\end{aligned}$$

3.3 Part c

The second correction to the energy of ψ_{23a} is given by the following. Note that only one term contributes because of how we defined our eigenstates in part a.

$$\begin{aligned}
E_{23a}^{(2)} &= \langle\psi_1|\, H_1 \,|\psi_{23a}\rangle / (\epsilon_{23} - \epsilon_1) \\
&= \left|\frac{1}{\sqrt{2}}(1,0,0)H_1(0,i,1)^T\right|^2 / (\epsilon_{23} - \epsilon_1) \\
&= (\Delta - \sigma)^2 / 2(\epsilon_{23} - \epsilon_1)
\end{aligned}$$