Final exam

Phys 137B, Fall 2016
(Dated: December 16, 2016)

Instructions: You are allowed two cheat sheets, handwritten on the front and back. Circle final results. Make your reasoning as clear as possible, because an unjustified answer will yield zero points even if it is correct. It is probably a good idea to read all problems first and start with the easiest. If you are mostly done with a problem except for "cosmetic" steps, do other problems first and come back later if you have time. If you run out of time, outline missing steps for partial credit. Likewise, if you find an error but run out of time, explain the error for partial credit. **Good luck!**

I. SCATTERING 1

A particle of mass m is scattered at a "soft sphere" potential

$$V(r) = \begin{cases} -V_0 & r < R, \\ 0 & r > R. \end{cases} \tag{1}$$

1. **5 points** Calculate the scattering amplitude $f(\theta)$ using the Born approximation.
2. **10 points** Calculate the $s-$ wave scattering amplitude using the partial wave method.

II. SCATTERING 2

5 points Show that the asymptotic solution

$$\phi(\vec{r}) = e^{ikz} + f(\theta)\frac{e^{ikr}}{r} \tag{2}$$

satisfies the Schrödinger equation, provided that $V(\vec{r}) \to 0$ faster than $1/r$ for $r \to \infty$.

III. ADIABATIC VS SUDDEN CHANGE

A particle of mass m is in the ground state of an infinitely high square-well potential with walls at $x = 0$ and $x = L$, i.e., $V = 0$ for $0 < x < L$ and $V = +\infty$ elsewhere. At a time $t = 0$, the right wall which was initially at $x = L$ starts moving right at a constant velocity v, expanding the potential well until it reaches $x = 2L$ at a time $t = T$.

- **5 points** Assuming that the process is adiabatic, what is the state $\psi(x,t)$ at $t = 0$ and $t = T$? Identify the dynamic phase and the Berry phase in your answer.
- **5 points** If, instead, the expansion happens in negligible time $t \ll T$, what is the (complex) amplitude that the particle remains in the ground state at $t = T$?

IV. 3-DIMENSIONAL WKB

In this problem, we are taking the first steps that extend the WKB method to three-dimensional problems. For a spherically symmetric potential $V(r)$, where r is the radial coordinate, find a way of estimating the energy of bound states using the WKB method.

1. **5 points** Make a separation of variables by writing the wave function ψ as $\frac{u(r)}{r}Y_l^m(\theta,\phi)$ and show that the differential equation for u is
$$u'' - \frac{2m}{\hbar^2}\left[\frac{l(l+1)\hbar^2}{2mr^2} + V(r) - E\right]u = 0$$
2. **5 points** Solve this radial equation by inserting $u(r) = a(r)e^{if(r)}$. Assuming that $V(r)$ varies slowly, find an explicit expression (involving an integral) for $u(r)$. Hint: it will be necessary to neglect a term in the differential equation, just as in the one-dimensional case.
3. **5 points** Use this solution and appropriate boundary conditions to calculate the energy eigenvalues in an three-dimensional, spherically symmetric potential well defined by
$$V(r) = \begin{cases} 0 & r < R, \\ +\infty & r \geq R. \end{cases}$$
When doing so, replace $l(l+1)$ with $(l+1/2)^2$. This is known as the *Langer correction* and makes the WKB result exact for many potentials. If you haven't solved part 2 of this problem, assume that this is a one-dimensional problem where $r = x$ and solve it.

V. TIME-DEPENDENT PERTURBATION

5 points A two-level system with states ψ_a and ψ_b, whose energy levels differ by $E_b - E_a = \hbar\omega_0$, is initially in the state ψ_a. It is then exposed to two short pulses,

$$V'(t) = V_0[\delta(t) + \delta(t-T)], \tag{3}$$

where $T > 0$ is a constant. Using first-order time-dependent perturbation theory, derive the probability that the system is in the state ψ_b after the second pulse. What choice of T maximizes the probability?

VI. THERMAL EXPANSION

A particle of mass m is in a Morse potential

$$U(x) = A(e^{-2\alpha x} - 2e^{-\alpha x}). \tag{4}$$

We use this as a model for the binding between the molecules in a solid. You may recall that $U(x)$ can be expanded as a Taylor series:

$$U(x) \approx A\left[-1 + \alpha^2 x^2 - \alpha^3 x^3 + \ldots\right]. \tag{5}$$

You may neglect higher-order corrections to this expansion for this problem.

1. **5 points** Without the x^3-term, show that the system is a harmonic oscillator having the usual states $|\psi_n\rangle$ with energies $E_n^0 = -A + \hbar\omega_0(n+1/2)$, where $\omega_0 = \alpha\sqrt{2A/m}$.

2. **5 points** Derive an expression for the perturbed states $\psi_n \approx \psi_n^0 + \psi_n^1$ in first-order perturbation theory and use it to calculate the expectation value $\langle\psi_n|x|\psi_n\rangle$. If the system is at a high temperature T, it is likely to be found in an excited state so that $\hbar\omega_0 n \sim k_B T$, where k_B is the Boltzmann constant. Express $\langle x\rangle$ as a function of the temperature T.

End of exam. There are six problems and a total of 60 points.

VII. EQUATIONS

A. Spherical Hankel and Bessel functions

A few spherical Hankel functions, $h_l^{\pm} = j_l \pm i n_l$:

$$h_0^+ = \frac{-i}{x}e^{ix}, \quad h_0^- = \frac{i}{x}e^{-ix}$$
$$h_1^+ = \left(\frac{-i}{x^2} - \frac{1}{x}\right)e^{ix}, \quad h_1^- = \left(\frac{i}{x^2} - \frac{1}{x}\right)e^{-ix}$$

They are exact solutions for the radial part of the Schrödinger equation at $V = 0$. For $x \to \infty$,

$$h_\ell^+ \to \frac{1}{x}(-i)^{\ell+1}e^{ix}, \quad h_\ell^- \to \frac{1}{x}i^{\ell+1}e^{-ix} \tag{6}$$

For small arguments $x \to 0$, we have

$$j_\ell \to \frac{2^\ell \ell!}{(2\ell+1)!}x^\ell, \quad n_\ell \to -\frac{(2\ell)!}{2^\ell \ell!}\frac{1}{x^{\ell+1}}. \tag{7}$$

B. Laplace operator in spherical coordinates

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \tag{8}$$

Spherical harmonics solve the angular differential equation:

$$\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{\partial^2 Y}{\partial\phi^2} = -l(l+1)\sin^2\theta Y. \tag{9}$$

C. An integral

$$\int\sqrt{a + \frac{b^2}{x^2}}\,dx = \xi + b\ln(x) - b\ln[b(b+\xi)], \tag{10}$$

where $\quad \xi = \sqrt{b^2 + ax^2}$.