Physics 137 B Spring 2012 - Final exam

PACS numbers:

Read the entire exam carefully and then do the easiest problem first. You do not need to insert numbers in any problem. If you know how to finish a problem but run out of time, describe the missing steps to obtain partial credit. If you obtain a result that you know is wrong (but don't have the time to correct) describe that you know it's wrong and why.

1. Free Dirac particle (40 points) Solve the Dirac equation

$$i\hbar\frac{\partial\Psi}{\partial t} = [c\vec{\alpha}\cdot\vec{p} + \beta mc^2]\Psi$$

for a free electron moving with momentum $p_z \equiv p$ along the z-axis. To do so, set $p_x = p_y = 0$, and insert $\Psi = e^{-iEt/\hbar}e^{ikz}u(k)$, where $u(k)$ is a 4-component vector that does not depend on the coordinates. Find an equation for $u(k)$ of the form

$$M(k)u(k) = \mathbb{1}E(k)u(k)$$

where $M(k)$ is a 4×4 matrix and $\mathbb{1}$ is the 4×4 unit matrix. Find $M(k)$ and find at least one solution of this equation by determining one eigenvalue $E(k)$ and the associated eigenvector u. Hint: you can guess the eigenvalues of the matrix from the relativistic energy of a particle of momentum $p = \hbar k$.

2. Ac Stark effect (20 points) Assume an atom with states $|n\rangle$, where $n = 1, 2, 3, \ldots.$ The states have energies E_n and are all nondegenerate. The atom is in the ground state $|1\rangle$ throughout this problem. The atom is in an oscillating electric field $\vec{E}(t) = \vec{E}_0\sin(\omega t)$ that causes a perturbation $H' = -e\vec{E}(t)\cdot\vec{r}$. The angular frequency ω is so low that it does not cause transitions between energy levels.

Use time-independent perturbation theory to calculate the energy level shift due to H'. Show that the first-order shift vanishes when averaged over one period of the oscillation, but that the time-average (again taken over one period of the oscillation) of the second-order shift can be nonzero. Show that this *ac Stark shift* has the form $\Delta E = E_0^2 A$ and derive an expression for the constant A. You do not need to evaluate the matrix elements $\vec{r}_{nm} = \langle n|\vec{r}|m\rangle$ of the position operator. For an atom in the ground state, is $A > 0$ or $A < 0$?

3. Shaken oscillator

(a) (20 points) A particle is confined near a minimum of a 1-dimensional potential $V(z) = V_0\sin^2(kz)$, where $V_0 > 0$ and k are constants and z is the coordinate. Show that this potential is approximately harmonic $V \approx \frac{1}{2}m\omega_0^2 z^2$ near $z = 0$ and calculate ω_0.

(b) (20 points) The atom is in the lowest eigenstate of the above potential. A truck speeding by the physics building causes vibrations, so that $V \approx \frac{1}{2}m\omega^2(z-z_0)^2$, where $z_0 = \epsilon k\sin\Omega t$ for $0 < t < T$ and $z_0 = 0$ otherwise. ϵ is a small dimensionless number. Use time-dependent perturbation theory to calculate the probability of transition to the first excited state to first order. You may assume that Ω is close to angular frequency ω_0 determined in part b.

4. Born approximation Using the Born approximation, calculate the differential cross section $d\sigma/d\Omega$ for slow particles ($\vec{k}\approx 0$) for

(a) (10 points) a cone-shaped potential $V(r) = V_0(r-R)/R$ inside a radius R, zero otherwise.

(b) (10 points) a soft-sphere potential $V(r) = V_0$ for $r < R$, zero otherwise

(c) (10 points) a spherical-shell potential $V(r) = V_0 R\delta(r-R)$.

5. Sudden release of a confined particle. In a one-dimensional problem, a particle of mass m is initially in the ground state of an harmonic oscillator with resonance frequency ω.

(a) (20 points) If $\Delta x \equiv \langle x^2\rangle - \langle x\rangle^2$ and $\Delta p \equiv \langle p^2\rangle - \langle p\rangle^2$, calculate the uncertainty product $\Delta x\Delta p$ from the particle's wave function $\psi_0(x) = (\alpha/\pi)^{1/4}e^{-\alpha x^2/2}$, where $\alpha = m\omega/\hbar$.

(b) (20 points) The eigenstate $\Psi_0(x, t = 0) = \psi_0(x)$ at time $t = 0$ can be written as a superposition of plane waves e^{ikx},

$$\Psi_0(x, t = 0) = \int_{-\infty}^{\infty} \phi(k) e^{ikx} dk.$$

Show that $\phi(k) = \beta e^{-k^2/2\alpha}$ and determine the constant β. If you don't want to do this problem, you can leave β undetermined for part (c).

(c) (30 points) Now suppose that at $t = 0$, the harmonic oscillator potential is suddenly switched off. Calculate the state for a time $t > 0$ after the switch off. For doing so, note that every plane wave e^{ikx} in the above integral acquires a phase factor $e^{-iE_k t/\hbar}$, where $E_k = \hbar^2 k^2/2m$.

Formulas. For some of the problems, the integral

$$\int_{-\infty}^{\infty} e^{-\alpha x^2/2 + \beta x} dx = \sqrt{\frac{2\pi}{\alpha}} e^{\beta^2/(2\alpha)}$$

might be useful. α, β can be real or complex as long as $\mathrm{Re}(\alpha) > 0$. You will also need the Dirac matrices $\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}$ and $\beta = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$, where $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, and $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\mathbb{1}$ is the 2×2 unit matrix.