Final Exam

Phys 137B, Spring 2013
(Dated: May 14, 2013)

Hints: several problems may be solved without calculation, or very little calculation. If you choose to do so, justify your answer specifically by the principles/ theorems that you are following.

1. **Rabi problem** Here is the Rabi solution for a two-level system with a level splitting of $\hbar\omega_0$ driven by perturbation of the form $Ve^{i\omega t}$, where $V_{ab} = \langle\psi_a|V|\psi_b\rangle = V_{ba}$

$$c_a(t) = \left[\cos\left(\frac{\Omega t}{2}\right) + i\frac{\delta}{\Omega}\sin\left(\frac{\Omega t}{2}\right)\right]e^{-i\delta t/2},$$
$$c_b(t) = i\frac{\Omega_R}{\Omega}\sin\left(\frac{\Omega t}{2}\right)e^{i\delta t/2}, \tag{1}$$

where $\Omega = \sqrt{\Omega_R^2 + \delta^2}$, $\Omega_R = V_{ab}/\hbar$ is the Rabi frequency and $\delta = \omega_0 - \omega$ is the detuning.

a) **5 points** What are the initial conditions $c_a(0), c_b(0)$ of this solution?

b) **10 points** Find the solution for initial conditions $c_a(0) = 0, c_b(0) = 1$. Hint: you don't need to solve the differential equations for the two-level system from scratch. Instead, modify the above solution.

c) **10 points** Your friend wants to know the solution for initial conditions $c_a(0) = c_b(0) = 1/\sqrt{2}$. Can you help? Hint: again, you don't need to solve the differential equations for the two-level system from scratch.

2. **Hollow-sphere scaterring** Consider s-wave scattering at the potential of a hollow sphere

$$V(r) = \begin{cases} 0 & r < r_0 \\ q^2/(2m) & r_0 < r < 2r_0 \\ 0 & r > 2r_0 \end{cases}, \tag{2}$$

where $q > 0$ is a constant. Derive the s-wave scattering cross section σ_0.

a) **10 points** Find the general solution to the radial equation in the innermost region, $r < r_0$. It is unnecessary to normalize it.

b) **10 points** Find the general solution in the middle region $r_0 < r < 2r_0$, with two constants A and B. Find $C = A/B$ by matching boundary conditions at the inner boundary. In the rest of the problem, call this quantity C.

c) **10 points** Find the logarithmic derivative

$$L_0 = \frac{r}{u_0(r)}\frac{du_0(r)}{dr}\bigg|_{r=R} \tag{3}$$

at the outer boundary as function of C.

d) **10 points** Find the s-wave scattering cross section σ_0. Express it as function of L_0.

e) **10 points** Calculate the differential cross section $d\sigma/d\Omega$ for the same potential in the Born approximation for large particle energies.

3. **Small scatterer (20 points)** Consider a potential well

$$V(r) = \begin{cases} 0 & r > R \\ -V_0 & r \le R \end{cases} \tag{4}$$

where $V_0 > 0$. A particle of mass m and wavenumber k scatters. Assume that $R \to 0$ in such a way that $w \equiv R\sqrt{2mV_0}/\hbar$ is constant. In the partial wave method, show that of the partial scattering amplitudes

$$a_l = -\frac{1}{ik}\frac{L_l j_l(x) - x j_l'(x)}{L_l h_l^{(1)}(x) - x(h_l^{(1)})'(x)}, \quad x = kR \tag{5}$$

where L_l are the logarithmic derivatives at $r = R$, only a_0 is relevant and proportional to $1/k$.

4. **Geometric phase (10 points)** A particle of mass m is in the second excited state $n = 2$ of a harmonic oscillator potential $V = \frac{1}{2}m\omega^2x^2$. What is the geometric phase γ_2 that the state accumulates when ω is adiabatically ramped down to half its initial value?

I. SPECIAL FUNCTIONS

The spherical Bessel, Neumann, and Hankel functions of the first kind, respectively, are

$$j_0(x) = \frac{\sin x}{x} \tag{6}$$
$$n_0(x) = -\frac{\cos x}{x} \tag{7}$$
$$h_0^{(1)}(x) = j_0(x) + in_0(x) = -\frac{e^{ix}}{x} \tag{8}$$