Final exam

Phys 137B
(Dated: May 8, 2016)

I. FINITE NUCLEAR RADIUS

In treating hydrogen, one usually assumes that the nucleus is pointlike. As a better approximation, consider the nucleus as a homogenously charged sphere of radius R. (Remember Gauss' law from electrostatics: the potential of a spherically symmetric charge distribution at radius r is given assuming that all charge located inside the radius r is concentrated at the center.)

a. **[10 points.]** Show that the effects of the finite nuclear radius can be described by a perturbation

$$H' = \begin{cases} -\frac{e^2}{4\pi\epsilon_0 R}\left(\frac{3}{2} - \frac{R}{r} - \frac{r^2}{2R^2}\right) & r < R, \\ 0 & r > R. \end{cases} \tag{1}$$

of the usual hamiltonian of the hydrogen atom.

b. **[10 points.]** Calculate the change of the ground state energy. If you have to calculate an integral, simplify it by assuming that $R << a_0$. The ground state wave function is $\psi = e^{-r/a_0}/\sqrt{\pi a_0^3}$.

II. SCATTERING

Consider scattering at a spherical potential wall of the form

$$V(r) = V_0 \delta(r - a). \tag{2}$$

The ingoing particle has an energy E and a wavenumber $k^2 = 2mE/\hbar^2$.

a. **[10 points].** Derive the scattering cross section $d\sigma/d\Omega$ using the Born approximation.

b. **[10 points].** If $R_i(r)$ is the inner radial function (for $r < a$) and $R_o(r)$ the outer one, show that

$$R_i(a) = R_o(a), \quad R_i'(a) + \frac{\hbar^2}{2m} V_0 = r_o'(a) \tag{3}$$

are that boundary conditions at $r = a$.

c. **[10 points].** Using the results of (b), derive the scattering phase as

$$\tan\delta_l = \frac{v j_l(ka)}{k j_l'(ka) n_l(ka) + k j_l(ka) n_l'(ka) + v n_l(ka)} \tag{4}$$

where A is an unimportant constant and $v = \hbar^2 V_0/(2m)$

d. **[10 points].** Find the condition for a resonance in the scattering cross section, How large does the (total) scattering cross section σ grow at such a resonance?

e. **[10 points].** Does the scattering cross section obtained by the Born approximation have a resonance? If not, is the resonance physical? Why does it only show up in the partial wave theory?

III. ONE-SIDED HARMONIC OSCILLATOR

A one-sided harmonic oscillator is a particle in a potential

$$V(x) = \begin{cases} \infty & x < 0, \\ \frac{1}{2} m\omega^2 x^2 & x \geq 0 \end{cases}. \tag{5}$$

a. **[10 points.]** Use the WKB method to determine its energy eigenstates. The integral

$$\int_0^a \sqrt{a^2 - x^2}\, dx = \frac{\pi}{4} a^2 \tag{6}$$

($a > 0$) is probably useful.

b. **[10 points.]** Which of the exact energies $\hbar\omega(n + 1/2)$, where $n = 0, 1, 2, \ldots$ of the (usual) two-sided harmonic oscillator are also the exact energies of the one-sided harmonic oscillator, and why? Compare them with the WKB solution. Miraculously, the WKB method delivers the exact energies in this case.

IV. ATOMIC CLOCK

Consider a two level system with states $|a\rangle$ and $|b\rangle$ that have energies $E_a = 0$ and $E_b = \hbar\omega > 0$, respectively. The system is initially in the state a. From time $t = 0$ to $t = \tau$, a sinusoidal perturbation $H'_{ab} = V_{ab}\cos(\omega t)$ is applied.

a. Use time-dependent perturbation theory (to first order) and show that, when the perturbation is turned off at the time τ,

$$\begin{aligned} c_b(\tau) &= -\frac{V_{ba}}{\hbar}\left[\frac{e^{i(\omega_0+\omega)\tau} - 1}{\omega_0 + \omega} + \frac{e^{i(\omega_0-\omega)\tau} - 1}{\omega_0 - \omega}\right] \\ &\approx -\frac{V_{ba}}{\hbar}\frac{e^{i(\omega_0-\omega)\tau} - 1}{\omega_0 - \omega} \end{aligned} \tag{7}$$

where, in the second line, we have neglected the first term because it is smaller when $\omega_0 \approx \omega$. In the following, work only with this simplified version.

b. Now assume that the perturbation is not applied from $t = 0$ to $t = \tau$ but instead from $t = t_1$ to $t = t_2$. Show that, when the perturbation is turned off at $t = t_2$,

$$c_b(t_2) \approx -\frac{V_{ba}}{\hbar}\frac{e^{i(\omega_0-\omega)t_2} - e^{i(\omega_0-\omega)t_1}}{\omega_0 - \omega}. \tag{8}$$

c. Now consider that the perturbation is applied twice, first from $t = 0$ to τ and then, after an interval T, from $t = \tau + T$ to $t = 2\tau + T$. What is the probability that the system is in the state $|b\rangle$ after the entire sequence? Use time-dependent perturbation theory. (Your answer may look messy, e.g., involving three exponentials. You don't need to simplify it further.)

V. ANGULAR MOMENTUM

Calculate the following commutators:

a. $[L^2, L_x]$,

b. $[L_x, \vec{r}^2]$, (where $\vec{r} = (x, y, z)$ is the position operator), and

c. $[L_z, p_x]$.

d. A system is in an eigenstate $|l, m\rangle$ of $\vec{L}^2$ and L_z. calculate the expectation value $\langle L_x \rangle$ and its uncertainty $(\Delta L_x)^2$.

VI. EQUATIONS

The following equations may be useful

$$\sin^2 \alpha = \frac{\tan^2 \alpha}{1 + \tan^2 \alpha} \tag{9}$$