Physics 137 B Spring 2012 - Midterm 1

February 26, 2012

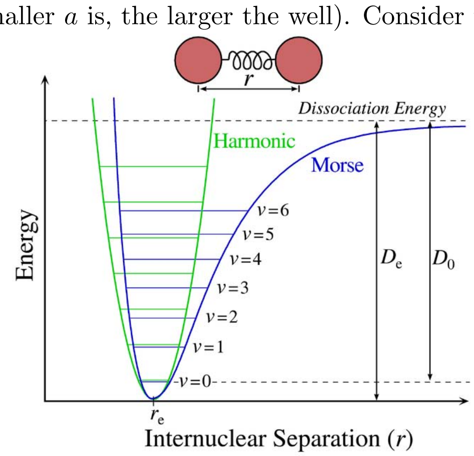
1. (10 points) An uncharged particle of mass m is placed in a gravitational potential $V(z) = mgz$ for $z > 0$, where $g = 9.8\,\mathrm{m/s}^2$ is the acceleration of free fall and z is the vertical coordinate. For $z < 0$, we assume $V(z) = +\infty$, *i.e.*, the particle has zero probability of getting to $z < 0$. Using the WKB method, find the energy levels of this system. Instruction: find the turning point z_E for a given energy E and then consider the boundary conditions at $z = 0$.

2. (10 points) A particle of mass m is enclosed in a one-dimensional box (square well potential) of length a:

$$U(x) = \begin{cases} 0 & 0 < x < a \\ \infty & \text{otherwise} \end{cases}$$

As you know, the normalized wave functions for this potential are $\psi_n(x) = \sqrt{2/a}\sin(n\pi x/a)$ and the energy levels are $E_n = (n\pi\hbar)^2/(2ma^2)$. Now consider a small modification of the potential by $V(x) = V_0 \sin(\pi x/a)$. Calculate the energy level shift in first order perturbation theory.

3. The potential between two atoms forming a molecule can be approximated by a Morse potential $U(r) = D_e(e^{-2a(r-r_e)} - 2e^{-a(r-r_e)})$ (see the figure), where r is the distance between the atoms, r_e is the distance at which the potential reaches its minimum of $-D_e$, and a controls the 'width' of the potential (the smaller a is, the larger the well). Consider this a one-dimensional problem.



1. (5 points) Consider oscillations of small amplitude around the minimum of the potential. What is the (angular) frequency ω_0 of the oscillations? The masses of the atoms are m_1 and m_2, respectively. What are the energy levels E_ν for a quantum number ν?

2. (5 points) Now consider the change of the ground state ($n = 0$) and first excited state ($n = 1$) energy, considering the fact that the potential is only approximately quadratic. Treat the difference of $U(r)$ and the harmonic approximation in first-order perturbation theory. The wave functions of the simple harmonic oscillator are $\psi_0(x) = (\alpha/\pi)^{1/4} e^{-y^2}$ and $\psi_1(x) = (\alpha/\pi)^{1/4}\sqrt{2} y e^{-y^2/2}$, where $\alpha = m\omega/\hbar$ and $y = \sqrt{\alpha}x$.

3. (10 points) Improve your estimate of the ground state energy by using the width parameter α of the ground-state wave function as a variational parameter. If you obtain a cubic equation for the value of α that minimizes the ground state energy, you may let it stand without solving it.

Integrals: We give you the following integrals:

$$\int \sqrt{1 - \frac{z}{z_E}}dz = -\frac{2}{3}\left(1 - \frac{z}{z_E}\right)^{3/2} z_E$$

$$\int_0^a \sin^2\left(\frac{n\pi x}{a}\right)\sin\left(\frac{\pi x}{a}\right)dx = \frac{4an^2}{(4n^2-1)\pi}$$

$$\int_{-\infty}^{\infty} x^n e^{-ax^2}dx = \begin{cases} a^{-(1+n)/2}\Gamma\left(\frac{1+n}{2}\right) & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$