1 Solution

Because the field is uniform and vertical (normal to Earth's surface), we infer that the Earth is approximated as a large, flat plane. By using Gauss's Law, one can find that a distance z above Earth's surface the magnitude of the electric field is

$$E(z) = \frac{\sigma}{2\epsilon_0}, \tag{1}$$

assuming the atmosphere to have vacuum permitivity. The direction is ambiguous, but if $\hat{E} = -\hat{z}$, the electric field vector points downward and therefore $\sigma < 0$. If $\hat{E} = \hat{z}$, then $\sigma > 0$. Without loss of generality, suppose $\hat{E} = \hat{z}$. Then, solving for σ and plugging in the necessary values up to one significant digit:

$$\sigma = E \cdot 2\epsilon_0 \tag{2}$$

$$\approx \left(1000\,\frac{\text{V}}{\text{m}}\right)\left(2 \cdot 9 \times 10^{-12}\,\frac{\text{F}}{\text{m}}\right) \tag{3}$$

$$\sigma \approx 2 \times 10^{-8}\,\text{C}\,\text{m}^{-2} \tag{4}$$

The energy density of the field is $u = \frac{\epsilon_0}{2}|E|^2$. Plugging in the necessary values and solving for one significant digit:

$$u \approx \frac{9 \times 10^{-12}}{2}\left(10^6\,\frac{\text{V}^2}{\text{m}^2}\right) \tag{5}$$

$$\approx 5 \times 10^{-6}\,\text{J}\,\text{m}^{-3} \tag{6}$$

2 Rubric

The problem is worth 20 points. These are divided as follows:

- **8 points:** Getting $E = \frac{\sigma}{2\epsilon_0}$.
- **1 point:** Solving for σ from the electric field equation.
- **1 point:** Simplifying numerically up to 1 significant digit.
- **8 points:** Knowing $u = \frac{\epsilon_0}{2}|E|^2$.
- **1 point:** Simplifying numerically up to 1 significant digit.
- **1 point (0.5 for each):** Correct numerical answers up to ± 1 significant digit.

Physics 7B Midterm 2 Problem 3 Solution

a. C_1 and C_2 are in parallel with one another. Thus, these two capacitors have an equivalent capacitance $C_{1,2} = C_1 + C_2$. This equivalent capacitance is in series with C_3, giving a total equivalent capacitance of:

$$\begin{aligned} C_{\text{eq},0} &= \left(\frac{1}{C_3} + \frac{1}{C_{1,2}}\right)^{-1} \\ &= \left(\frac{1}{C_3} + \frac{1}{C_1 + C_2}\right)^{-1}. \end{aligned} \tag{1}$$

b. Since this circuit is equivalent to a capacitor of capacitance $C_{\text{eq},0}$, the energy stored by the circuit is given by $U_0 = \frac{1}{2}C_{\text{eq},0}V_0^2$. Thus,

$$U_0 = \frac{1}{2}\left(\frac{1}{C_3} + \frac{1}{C_1 + C_2}\right)^{-1} V_0^2. \tag{2}$$

c. Continuing to treat the circuit as a single capacitor with equivalent capacitance $C_{\text{eq},0}$, the charge on this capacitor (and therefore on the terminals) would be $Q = C_{\text{eq},0}V_0$. Once C_3 is shorted, it is effectively replaced by a wire running from point a to point b. Thus, the new equivalent capacitance of the circuit is:

$$C_{\text{eq}} = C_1 + C_2. \tag{3}$$

Since the voltage source has been disconnected, all charge originally on the terminals must remain there. Thus, even after shorting C_3, the charge on the positively charged terminal is still Q. Thus, the voltage on the terminals after shorting C_3 will be:

$$\begin{aligned} V &= \frac{Q}{C_{\text{eq}}} \\ &= V_0 \frac{C_{\text{eq},0}}{C_{\text{eq}}} \\ &= V_0 \frac{\left(\frac{1}{C_3} + \frac{1}{C_1+C_2} \right)^{-1}}{C_1 + C_2} \\ &= \frac{V_0}{\frac{C_1+C_2}{C_3} + 1} \\ &= \frac{C_3 V_0}{C_1 + C_2 + C_3}. \end{aligned} \tag{4}$$

Midterm 2 Bale Rubric (Question 4)

20 points total

a)

Using Gauss's Law for r > R_3, one finds that E = $Q'/4\pi\varepsilon_0 r^2$ in the radial direction.

1 point for using Gauss's Law
1 point for saying that E is in the radial direction
2 points for correct magnitude of E

b)

Using Gauss's Law for R_2 < r < R_3, one finds that E = $(Q' - Q)/4\pi\varepsilon_0 r^2$ in the radial direction.

1 point for using Gauss's Law
1 point for saying that E is in the radial direction
2 points for correct magnitude of E

c)

Using Gauss's Law for R_1 < r < R_2, one finds that E = $-Q/4\pi\varepsilon_0 K r^2$ in the radial direction.

1 point for using Gauss's Law
1 point for saying that E is in the radial direction
3 points for correct magnitude of E

(2 points partial credit if correct magnitude but didn't factor in the dielectric, e.g. forgot to include dielectric constant K)

d)

Using Gauss's Law for r < R_1, one finds that E = 0.

1 point for using Gauss's Law
2 points for correct answer

e)

Matching the answers to parts b and c at r = R_2, one finds that $Q' = Q(K-1)/K$.

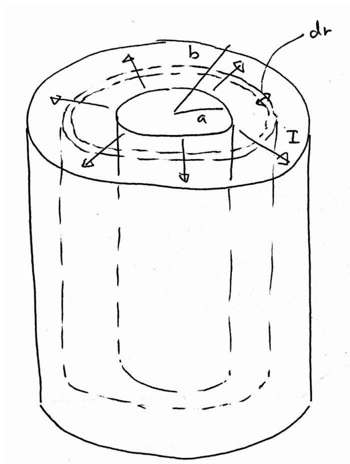
2 points for a method involving matching solutions at $r = R_2$
2 points for correct answer

(1 point for incorrect method)
(3 points for correct method but incorrect solution due to propagated error from earlier part)

Problem *5*

20 pts

We can treat sea water between two coaxial cylinders as a resistor with resistivity ρ in the shape of a thick cylindrical shell of inner radius a, outer radius b, and length L. We are asked to calculate the resistance between the cylinders, so we can assume that current flows radially outward (it is not running along the length).



Now we can decompose this resistor as resistors (thin cylindrical shells) connected in series, cylindrical with radius $a \le r \le b$, length L, and thickness dr. Eash shell has the resistance

$$dR = \rho \frac{dr}{A} = \rho \cdot \frac{dr}{2\pi r L}$$

Then, we integrate along the radial direction to get R.

$$R = \int dR = \int_a^b \rho \cdot \frac{dr}{2\pi r L} = \frac{\rho}{2\pi L} \int_a^b \frac{dr}{r} = \frac{\rho}{2\pi L} \ln(b/a).$$

Grading Rubric:

+3 Show $R = \rho L/A$
+2 Show $R = dR$
+5 Show $dR = \rho \frac{dr}{A}$
+8 Show $R = \int dR = \int_a^b \rho \cdot \frac{dr}{2\pi r L}$
(-5 Assuming current flows along the length)
(-2 Minor mistakes - e.g. having ln(a/b), which is negative.)
+2 Correct answer