Final Exam – Multivariable Calculus

Math 53, December 19, 2009. Instructor: E. Frenkel

You have to answer the following 11 equally-weighted questions. **On each problem, THERE SHOULD BE ENOUGH WORK SHOWN to justify your answer. Please encircle your final answers.** Don't forget to write your name and the name of your TA on this page.

You are not allowed to use any materials or devices during the exam except for a list of formulas written by yourself on ONE standard size sheet of paper.

No calculators are allowed during the exam.

YOUR NAME: JAC

YOUR TA'S NAME: ______________________

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1. For each statement below, determine whether it is true or not. Circle T if it is true, or F if it is false.

(1) If $\mathbf{F} = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$ is a conservative vector field on the plane, then $P_y = Q_x$.

T F

(2) If the line integral of a vector field $\mathbf{F}$ along any closed curve is equal to 0, then $\mathbf{F}$ is conservative.

T F

(3) There exists a vector field $\mathbf{F}$, such that $\operatorname{curl} \mathbf{F} = x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}$.

T F

(4) There exists a vector field $\mathbf{F}$, such that $\operatorname{div} \mathbf{F} = x^3 + y^3 + z^3$.

T F

(5) The line integral of the vector field $x\mathbf{i}$ over any closed curve in $\mathbb{R}^3$ equals 0.

T F

(6) If a vector field $\mathbf{F}$ is well-defined on the entire $\mathbb{R}^3$ and $\operatorname{curl} \mathbf{F} = 0$, then there exists a function f such that $\mathbf{F} = \nabla f$.

T F

(7) The flux of any vector field across any closed surface in $\mathbb{R}^3$ equals 0.

T F

(8) Any surface in $\mathbb{R}^3$ has two distinct orientations.

T F

(9) Any curve in $\mathbb{R}^3$ has two distinct orientations.

T F

(10) If $\mathbf{F}$ is a vector field well-defined on the entire $\mathbb{R}^3$ and such that $\operatorname{div} \mathbf{F} = 0$, then the flux of $\mathbf{F}$ across any surface in $\mathbb{R}^3$ equals 0.

T F

2. Find the mass of a wire in the shape of the helix $x = t, y = \cos t, z = \sin t, 0 \leq t \leq 2\pi$, if the mass density function at any point is equal to the square of the distance from the origin.

3. The force exerted by an electric charge at the origin on an electron at a point (x, y, z) with position vector $\mathbf{r} = \langle x, y, z \rangle$ is $\mathbf{F}(\mathbf{r}) = -K\mathbf{r}/|\mathbf{r}|^3$, where K is a constant. Find the work done by this force as the electron moves along the straight line segment from $(2, 0, 0)$ to $(2, 1, 5)$.

4. (a) Determine whether the vector field

$$\mathbf{F}(x,y) = \left(2x - \frac{1}{y}\sin\frac{x}{y}\right)\mathbf{i} + \frac{x}{y^2}\sin\frac{x}{y}\mathbf{j}$$

is conservative or not on the region $\{(x,y)|y > 0\}$ of the plane. If it is, find all functions $f(x,y)$ such that $\mathbf{F} = \nabla f$.

(b) Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}$ is as above and C is the line segment connecting the points $A = (\pi, 1)$ and $B = (0, 5)$ and oriented from A to B.

5. Evaluate the surface integral $\iint_S z dS$, where S is the part of the sphere $x^2+y^2+z^2=4$ that lies inside the cylinder $x^2+y^2=1$ and above the xy plane.

6. Evaluate the flux $\iint_T \mathbf{F} \cdot d\mathbf{S}$ of the vector field $\mathbf{F}(x, y, z) = -x\mathbf{i} + xy\mathbf{j} + zx\mathbf{k}$ across the triangle T with vertices $(1, 0, 0), (0, 2, 0)$, and $(0, 0, 2)$, with downward orientation.

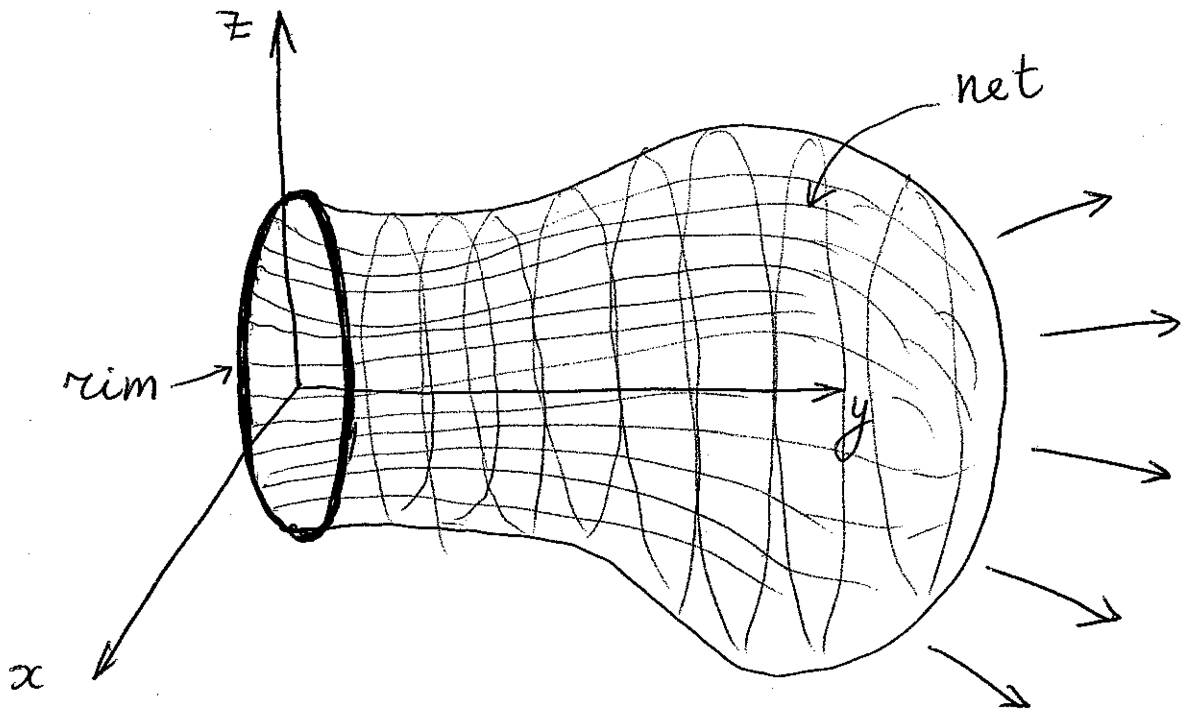
7. Evaluate the line integral $\int_C (xe^{-2x}dx + (y^4 + xy^2 + x^2)dy)$, where C consists of two circles: $x^2 + y^2 = 1$, oriented clockwise, and $x^2 + y^2 = 4$, oriented counterclockwise.

8. Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where

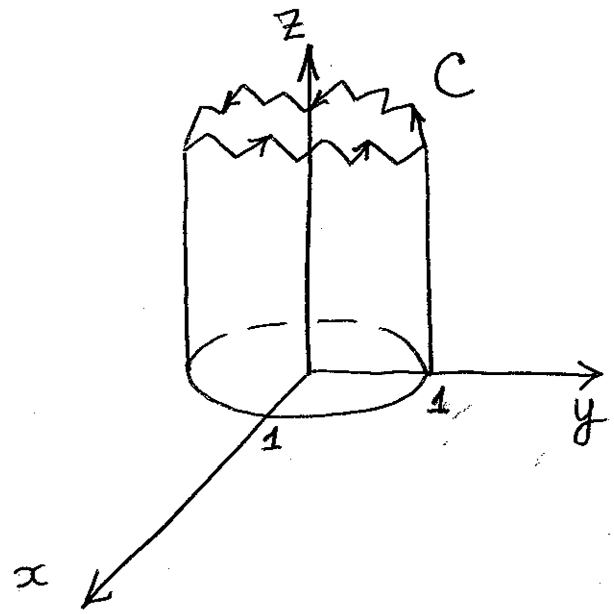
$$\mathbf{F} = (z^2x + e^{z^2-y^2})\mathbf{i} + (\frac{1}{3}y^3 + x^2y + \sin(z + x^2))\mathbf{j} + x^2\mathbf{k}$$

and S is the top half of the sphere $x^2 + y^2 + z^2 = 1$ oriented upward.

9. A fisherman's net has a rim, which is a circle of radius 5. He fixes it in the sea in such a way that the rim is in the xz plane with the center at the origin. We don't know the exact position of the net, but we know that the velocity of the water is given by the vector field $\mathbf{F} = (x^4 + 2y^2)\mathbf{i} + (3 - y^2)\mathbf{j} + (2yz - 4x^3z)\mathbf{k}$. Find the flux of this vector field through the net, oriented as shown on the picture.



10. A broken wine bottle is placed on the xy plane as shown on the picture. It consists of a portion of a cylinder of radius 1 along the z axis, and its bottom is the unit disc in the xy plane centered at the origin. Let C be the path along the broken edge oriented as shown on the picture, and let $\mathbf{F} = -y\mathbf{i} + 2x\mathbf{j} + 10z\mathbf{k}$. Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$.



11. Let S_r denote the sphere of radius r with the center at the origin, with outward orientation. Suppose that $\mathbf{E}$ is a vector field well-defined on all of $\mathbb{R}^3$ and such that

$$\iint_{S_r} \mathbf{E} \cdot d\mathbf{S} = ar + b,$$

for some fixed constants a and b.

(a) Compute in terms of a and b the following integral:

$$\iiint_D \operatorname{div} \mathbf{E}\, dV,$$

where $D = \{(x, y, z) | 25 \leq x^2 + y^2 + z^2 \leq 49\}$.

(b) Suppose that in the above situation $\mathbf{E} = \operatorname{curl} \mathbf{F}$ for some vector field $\mathbf{F}$. What conditions, if any, does this place on the constants a and b?