Problem 1 (30 points)

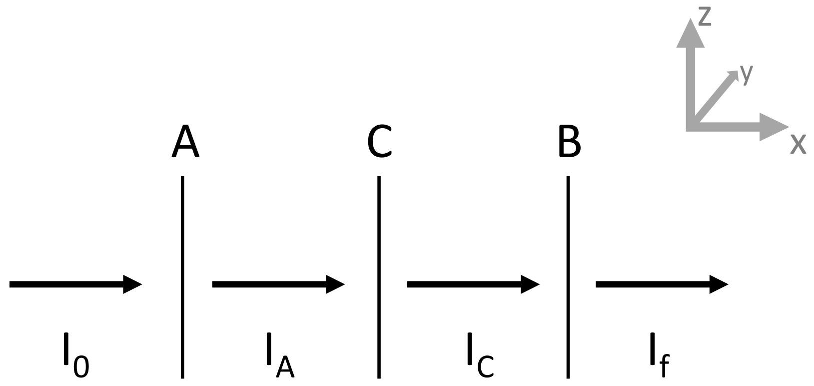


Figure 1: View of polaroid placement, looking in $+\hat{y}$-direction, part (a).

(a) After the initial unpolarized light passes through polaroid A, the intensity will reduce by half:

$$I_A = \frac{I_0}{2}, \tag{1}$$

and the light between A and C will be polarized vertically (along the $\hat{z}$-direction).
If the polarization axis of C is at angle θ_C to the polarization axis of A, the intensity I_C will be:

$$I_C = I_A \cos^2\theta_C = \frac{I_0}{2}\cos^2\theta_C. \tag{2}$$

Since the angle between the axes of A and B is $\frac{\pi}{2}$, the angle between C and B is $\frac{\pi}{2} - \theta_C$, and

$$\frac{I_f}{I_0} = \frac{I_C\cos^2(\frac{\pi}{2}-\theta_C)}{I_0} = \boxed{\frac{1}{2}\cos^2(\theta_C)\cos^2(\frac{\pi}{2}-\theta_C)} \tag{3}$$

You can notice that $\cos(\frac{\pi}{2}-\theta) = \sin\theta$ and use a double angle identity to say that

$$\frac{I_f}{I_0} = \frac{1}{2}\cos^2(\theta_C)\cos^2(\frac{\pi}{2}-\theta_C) = \frac{1}{2}\cos^2\theta_C\sin^2\theta_C = \frac{1}{2}\left(\frac{\sin(2\theta_C)}{2}\right)^2. \tag{4}$$

Since the sin function is maximal when its argument is $\frac{\pi}{2}$, the maximum for $\frac{I_f}{I_0}$ occurs when $2\theta_C = \frac{\pi}{2}$, or $\boxed{\theta_C^* = \frac{\pi}{4}}$.

The value for θ_C^* can also be found by setting the derivative of I_f/I_0 (with respect to θ_C) equal to zero.

$$\frac{d(I_f/I_0)}{d\theta_C} = 0 = \cos\theta_C\sin\theta_C(\cos^2\theta_C - \sin^2\theta_C) \tag{5}$$

gives minima at $\theta_C = \frac{\pi}{2}$ and $\theta_C = 0$ (for $\cos\theta_C = 0$ and $\sin\theta_C = 0$), and a maximum at $\theta_C = \frac{\pi}{4}$ (when $\cos^2\theta_C - \sin^2\theta_C = 2\cos^2\theta_C - 1 = 0$).

(b) If $\theta_C^* = \frac{\pi}{4}$, then

$$\begin{aligned}\frac{I_f}{I_0} &= \frac{1}{2}\cos^2(\theta_C^*)\cos^2(\frac{\pi}{2}-\theta_C^*) = \frac{1}{2}\cos^2\frac{\pi}{4}\cos^2\frac{\pi}{4} \\ &= \frac{1}{2}\left(\frac{1}{\sqrt{2}}\right)^2\left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) \\ &= \boxed{\frac{1}{8}}\end{aligned} \tag{6}$$

Problem 1 (30 points), continued

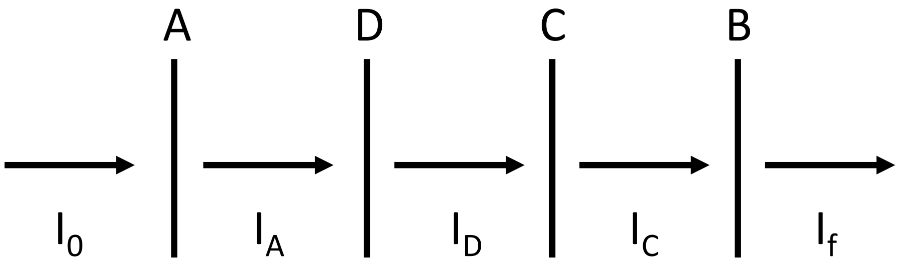


Figure 2: View of polaroid placement, looking in $+\hat{y}$-direction, part (c).

(c) The final intensity for the four polaroid setup will be

$$I_f = \frac{I_0}{2}\cos^2\theta_{AD}\cos^2\theta_{DC}\cos^2\theta_{CB} \tag{7}$$

where θ_{xy} is the angle between the polarization axes of polaroids x and y.
Since D is placed so that its axis is at an angle $\frac{\theta_c^*}{2}$ to the vertical (and to the axis of A), $\theta_{AD} = \frac{\theta_c^*}{2}$.
Since C is at angle θ_C^* to the vertical, $\theta_{DC} = \theta_{AC} - \theta_{AD}$, or $\theta_{DC} = \frac{\theta_C^*}{2}$.
And the angle between C and B is still $\theta_{CB} = \frac{\pi}{2} - \theta_C^*$.

$$\boxed{\frac{I_f}{I_0} = \frac{1}{2}\cos^4\left(\frac{\theta_C^*}{2}\right)\cos^2\left(\frac{\pi}{2} - \theta_C^*\right)} \tag{8}$$

To get the above expression in terms of only $\sin\theta_C^*$ and $\cos\theta_C^*$, we can use a half angle identity $\cos\frac{\theta}{2} = \sqrt{\frac{\cos\theta+1}{2}}$:

$$\frac{I_f}{I_0} = \frac{1}{2}\left(\frac{\cos\theta_C^* + 1}{2}\right)^2 \sin^2\theta_C^*. \tag{9}$$

And we can plug in our answer to (a), $\theta_C^* = \frac{\pi}{4}$ to obtain our result:

$$\begin{aligned} \frac{I_f}{I_0} &= \frac{1}{2}\left(\frac{\cos\theta_C^* + 1}{2}\right)^2 \sin^2\theta_C^* \\ &= \frac{1}{2}\left(\frac{\frac{1}{\sqrt{2}} + 1}{2}\right)^2\left(\frac{1}{2}\right) = \frac{1}{4}\left(\frac{(\frac{1}{\sqrt{2}} + 1)\sqrt{2}}{2\sqrt{2}}\right)^2 \\ &= \boxed{\frac{1}{8}\left(\frac{1+\sqrt{2}}{2}\right)^2} \end{aligned} \tag{10}$$

The boxed expression above is written this way to highlight something: the final intensity after adding polaroid D is *greater* than the three polaroid configuration used in parts (a) and (b). The part in parentheses will be greater than 1, leading to $\boxed{\frac{I_f}{I_0} > \frac{1}{8}}$, while our answer to (b) was $\frac{I_f}{I_0} = \frac{1}{8}$. The point is also illustrated by seeing that $\cos^4(\frac{\theta}{2}) > \cos^2\theta$ when $0 < |\theta| \leq \frac{\pi}{2}$

Problem 2 Solution

April 13, 2017

Part (a) indicates that diffraction effects should be ignored. The light shines straight through the hole to the back of the box, where it creates a spot the same size as the hole itself, so $S = W$.

Part (b) asks us to develop a correction to the answer from part (a) due to diffraction. We can write the correction to the spot size as a Taylor series in the non-dimensional parameter $\frac{\lambda}{W}$

$$S(W) \approx W + C_1 \left(\frac{\lambda}{W}\right) + C_2 \left(\frac{\lambda}{W}\right)^2 + ...$$

where the coefficients C_n are to be determined. In the case where the hole size W is much, much larger than the wavelength λ, we can recover our answer from part (a). Assuming the correction is very small, we can ignore everything but the linear term, and our job is simply to choose a value for C_1. Recall that the Rayleight criterion stipulates that the "fuzziness" of a point source due to diffraction obeys the relation $\sin\theta \approx 1.22 \left(\frac{\lambda}{W/2}\right)$. This suggests we use the approximation

$$S(W) \approx W + 2.44\, L \left(\frac{\lambda}{W}\right) \quad .$$

We can see from this approximation that if the hole size is made very small, diffraction dominates and the spot size is made very large. We can see also that if the hole size is made very large, diffraction can be neglected and the spot size, which is essentially the size of the hole, is also very large. We can find the intermediate hole size for which the spot size is smallest by minimizing the above approximation. The minimal spot size occurs when $W \approx \sqrt{2.44\lambda L}$.

Very few people received full credit on this problem. The most common mistake by far was to use the Rayleigh criterion naively in part (b), and to neglect the constant term W. Following this logic, most people said that W should be made as large as possible (i.e. $W = L$) in order to minimize the spot size. This, however, cannot be true— it's the statement that a box with a hole that takes up its entire side will minimize spot size, which intuition and everyday experience would indicate is impossible.

(a) Using the relativistic velocity addition formulas we find:
(Note: galilean methods are not enough to recieve full credit)

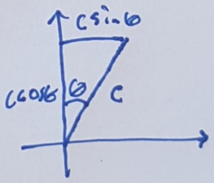
$$u_{\parallel} = \frac{u'_{\parallel} + v}{1 + \frac{v \cdot u_{\parallel}}{c^2}} \quad \text{and} \quad u_{\perp} = \frac{\sqrt{1-\frac{v^2}{c^2}}\, u'_{\perp}}{1 + \frac{v \cdot u'_{\parallel}}{c^2}} = \frac{1}{\gamma} \frac{u'_{\perp}}{1 + \frac{v \cdot u'_{\parallel}}{c^2}}$$

for us in the x and y directions gives:

$$u'_x = \frac{u_x - v}{1 - \frac{v}{c^2} u_x} \quad \text{and} \quad u'_y = \frac{1}{\gamma} \frac{u_y}{1 - \frac{v u_x}{c^2}}$$

since $u'_x = 0$

$$\Rightarrow \quad 0 = \frac{u_x - v}{1 - \frac{v}{c^2} u_x} \Rightarrow \boxed{u_x = v}$$



However $u_x = c \sin\theta$
and $u_y = c \cos\theta$

$$\Rightarrow \boxed{v = c \sin\theta \text{ in the negative x-direction}}$$

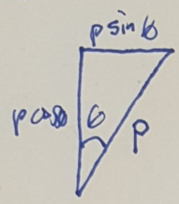
Notice if we look at u'_y we find

$$u'_y = \sqrt{1 - \frac{v^2}{c^2}} \cdot \frac{u_y}{1 - \frac{v^2}{c^2}} = \frac{u_y}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{c \cdot \cos\theta}{\sqrt{1 - \sin^2\theta}} = \frac{c \cdot \cos\theta}{\sqrt{\cos^2\theta}} =$$

Thus the speed of the photon is observed to be c in S'

(b) There are three main approaches to solving this problem.

(i) Using relativistic 4 momentum we find



$$\begin{pmatrix} p_0' \\ p_x' \\ p_y' \\ 0 \end{pmatrix} \text{ in } S' \qquad \begin{pmatrix} p_0 \\ p_x \\ p_y \\ 0 \end{pmatrix} \text{ in } S$$

first thing to recognize is that since the boost is in the x-direction

$p_y' = p_y$ remains unchanged

also note that since this is a photon $E = p_0 = pc$ (3 momentum)

since $p_x' = 0 \Rightarrow p_0' = p_y'$

but in S $\quad p_y = p\cos\theta$

$$\Rightarrow \quad p_0' = p_y' = p_y = p\cos\theta$$

$$\Rightarrow \quad \boxed{E' = E\cos\theta}$$

Since the energy of a photon is proportional to f $\quad E \propto f$

$$\Rightarrow \quad \boxed{f' = f\cos\theta}$$

Note this implies a redshift, as expected.

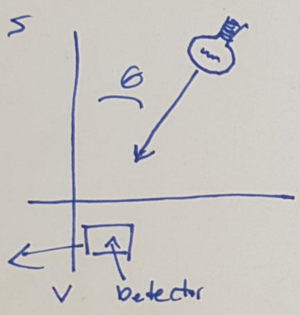
(ii) $E' = \gamma(E - \beta p_x c)$ when β and p_x are in the same direction

$$\Rightarrow E' = \gamma(E - \beta pc \cdot \sin\theta) = \gamma E(1 - \beta\sin\theta) = E\frac{(1-\beta\sin\theta)}{\sqrt{1-\beta^2}}$$

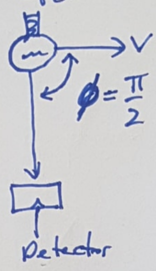
however $\beta = \sin\theta \Rightarrow E' = E\frac{(1-\sin^2\theta)}{\sqrt{1-\sin^2\theta}} = E = \sqrt{\cos^2\theta} = E\cos\theta$

since $E \propto f \Rightarrow$ $\boxed{f' = f\cos\theta}$

(iii) S



S' Detector frame



$\phi = \frac{\pi}{2}$ the angle between light emission an velocity of object.

$$f' = \frac{f}{\gamma(1+\frac{v}{c}\cos\phi)} = \frac{f}{\gamma}$$ which is simply the transverse doppler Effect

(relativistic doppler Effect)

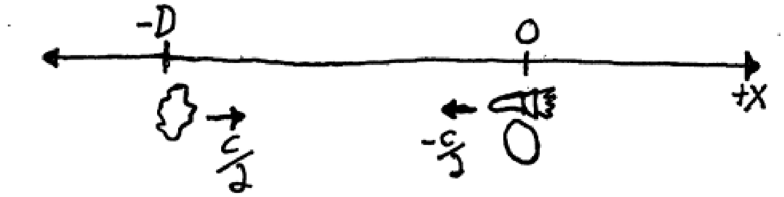
$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1-\sin^2\theta}} = \frac{1}{\cos\theta}$$

$\Rightarrow$ $\boxed{f' = f\cos\theta}$

Problem 4

Part a

For this problem, we will set up coordinates as follows:



To get the velocity of the asteroid in the rocket's frame, we use the velocity addition formula

$$u'_x = \frac{u_x - V}{1 - \frac{u_x V}{c^2}}$$

putting in $u_x =$ c/2 and $V = -c/2$, we find

$$u'_x = \frac{4c}{5}$$

Part b

There are a couple of different ways to solve this problem correctly.

Method 1

One can find the coordinates of the crash in the Earth frame and then perform a Lorentz transformation to get the time coordinate of the crash in the rocket's frame. From the symmetry of the problem, we see that the the spatial coordinate of the crash in the Earth frame is $x_c = -\frac{D}{2}$. So the time coordinate is simply $t_c = \frac{x_c}{v_R} = \frac{D/2}{c/2} = \frac{D}{c}$. Thus, the the rocket frame, the crash occurs at

$$\begin{aligned} ct'_c &= \gamma\left(ct_c - \beta x_c\right) \\ &= \frac{2}{\sqrt{3}}\left(D - \frac{D}{4}\right) \\ &= \frac{\sqrt{3}}{2}D \end{aligned}$$

Method 2

The second method involves using time dilation. Most students that got this problem correct used this method. One can simply use

$$t_c = \gamma t'_c$$

Using the above expession for t_c, we again find that $t'_c = \frac{\sqrt{3}D}{2c}$. The reason this method works is that the inverse Lorentz transformation reads

$$ct_c = \gamma\left(ct'_c + \beta x'_c\right)$$

In the rocket frame, the crash occurs at $x'_c = 0$. Hence the above equation reduces to the usual time dilation.

Method 3

This method was not really used, but I will write it out for your enjoyment. The idea is to find the time of the crash in the rocket frame by doing kinematics in the rocket frame. From part (a), we know that the asteroid moves with velocity $4c/5$ in the rocket frame. We then model the path of the asteroid as

$$x'_A(t') = \frac{4c}{5}t' + C$$

where $x'_A(t')$ is the position of the asteroid in the rocket frame at rocket time t'. To find the constant C, we transform the initial coordinates of the asteroid in the Earth frame to the rocket frame:

$$\begin{aligned} ct'_{A_0} &= \gamma\left(ct_{A_0} - \beta x_{A_0}\right) = \frac{2}{\sqrt{3}}\left(0 + \frac{D}{2}\right) = -\frac{D}{\sqrt{3}} \\ x'_{A_0} &= -\frac{2}{\sqrt{3}}D \end{aligned}$$

Then we require $x'_A(t'_{A_0}) = x'_{A_0}$ so that $C = \frac{-6D}{5\sqrt{3}}$. Then to find the time coordinate of the crash we set

$$x'_a(t'_c) = 0 = \frac{4c}{5}t'_c - \frac{6D}{5\sqrt{3}}$$

and we obtain the same answer as above.

Incorrect Method

Many students used the incorrect method of length contraction to get an answer. With this approach, one find

$$t'_c = \frac{D'}{\frac{4c}{5}} = \frac{D}{\gamma}\frac{5}{4c} = \frac{5\sqrt{3}D}{8c}$$

Clearly this does not yield the correct answer. That this approach fails is evident from the equation for $x'_a(t_c)$ in Method 3 above. At $t' = 0$, the asteroid is located at $x' = \frac{-6D}{5\sqrt{3}}$, not the length contracted distance $\frac{D}{\gamma} = \frac{\sqrt{3}D}{2}$. The physical reason why you cannont simply use length contraction is that there is no fixed length to contract since the asteroid is moving relative to the rocket.