EE 120: Signals & Systems
Department of Electrical Engineering and Computer Sciences — Midterm Exam #1
UNIVERSITY OF CALIFORNIA, BERKELEY — February 29, 2016, 2:10-4:00pm

Instructions:

- There are **five** questions on this midterm. Answer each question in the space provided, and **clearly label the parts of your answer**. You can use the additional blank pages at the end for scratch paper if necessary. **Do NOT write answers on the back of any sheet or in the additional blank pages. Any such writing will NOT be graded.**
- Each problem is worth 20 points, and you may solve the problems in any order.
- Show all work. If you are asked to prove something specific, you must give a derivation and not quote a fact from your notes sheet. Otherwise, you may freely use facts and properties derived in class; just be clear about what you are doing!
- None of the questions requires a very long answer, so avoid writing too much! Unclear or long-winded solutions may be penalized.
- You may use one double-sided sheet of notes. **No calculators are allowed** (or needed).
- The following formulas may (or may not) be useful:

$$\int_{-\infty}^{\infty} e^{-(x-b)^2}\,dx = \sqrt{\pi} \qquad \text{for real or imaginary } b \text{, and}$$

$$\int u\,dv = uv - \int v\,du \qquad \text{(integration by parts).}$$

Your Name:

Your Student ID:

Name of Student on Your Left:

Name of Student on Your Right:

For official use – do not write below this line!

Q1	Q2	Q3	Q4	Q5	Total

Problem 1. (*Solving a Mystery*) Consider a discrete-time LTI system with frequency response $H(e^{j\omega})$ and corresponding impulse response $h[n]$.

a) Suppose you are given the following three clues about the system:

   i) The system is causal.

   ii) $H(e^{j\omega}) = H^*(e^{-j\omega})$.

   iii) The DTFT of the sequence $h[n+1]$ is real.

   Show that the system is FIR.

b) Suppose you are given two more clues:

   iv) $\frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})d\omega = 2$.

   v) $H(e^{j\pi}) = 0$.

   Is there enough information to identify the system uniquely? If so, determine the impulse response $h[n]$. If not, specify as much as you can about the sequence $h[n]$.

(Additional space for Problem 1)

Problem 2. (*Discrete-time Convolution*) Consider the three sequences

$$\begin{aligned} v[n] &= u[n] - u[n-6] \\ w[n] &= \delta[n] + 2\delta[n-2] + \delta[n-4] \\ q[n] &= v[n] * w[n]. \end{aligned}$$

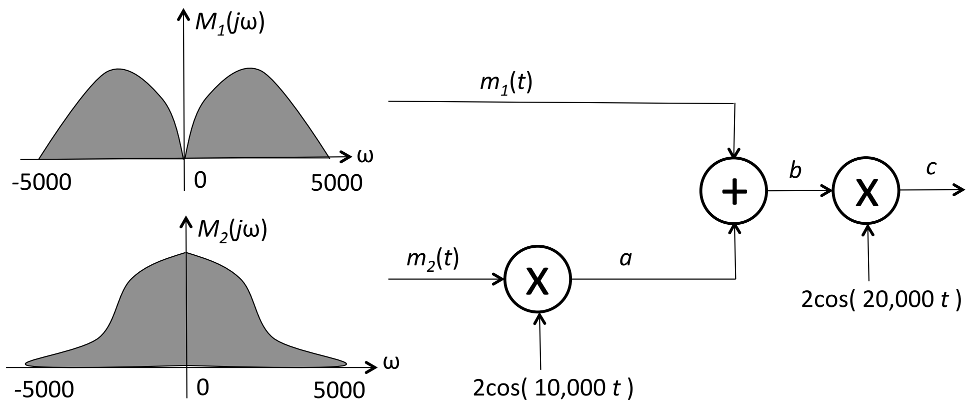
a) Find and sketch the sequence $q[n]$.

b) Find and sketch the sequence $r[n]$ such that $r[n] * v[n] = \sum_{k=-\infty}^{n-1} q[k]$.

c) Is $q[-n] = v[-n] * w[-n]$? Justify your answer.

(Additional space for Problem 2)

Problem 3. (*Multi-user Communications*) The illustration below shows a modified amplitude modulation scheme to simultaneously transmit two signals $m_1(t)$ and $m_2(t)$ over a wireless channel.



a) Is the system shown in the figure above linear? Is it time invariant? Is it BIBO stable?

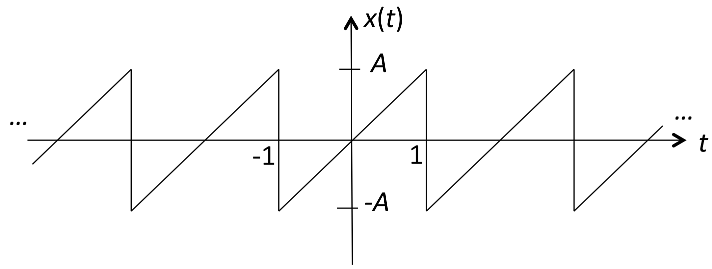
b) Given that the respective spectra of $m_1(t)$ and $m_2(t)$ are $M_1(j\omega)$ and $M_2(j\omega)$ as illustrated, sketch the spectra at points a, b and c in the system. Carefully label all points on the ω-axis.

c) Design a receiver to separately recover signals $m_1(t)$ and $m_2(t)$ from the modulated signal at point c. Explicitly state the frequency response and impulse response of any filters you use (your filters may be ideal).

(Additional space for Problem 3)

Problem 4. (*Fun with Fourier Series*)

a) Consider the signal $x(t)$ of period $T = 2$, shown below.



Compute the Fourier Series coefficients $a_k, k \in \mathbb{Z}$ for the signal $x(t)$.

b) A real signal $x(t)$ with fundamental period T is said to possess *half-wave symmetry* if it satisfies $x(t) = -x(t - T/2)$. That is, half of one period of the signal is the negative of the other half. Find the Fourier Series coefficients for even harmonics (i.e., a_{2k}, for $k \in \mathbb{Z}$) for all periodic signals with half-wave symmetry.

c) A real periodic signal $x(t)$ with half-wave symmetry is input to an LTI system with frequency response $H(j\omega)$. Does the corresponding output $y(t)$ also have half-wave symmetry? If yes, prove it. If not, provide a counterexample.

(Additional space for Problem 4)

Problem 5. (*Heat Flow*) The *heat equation* is a famous partial differential equation. Joseph Fourier invented the Fourier transform to solve it in 1822!

The basic setup goes like this. Think of a one-dimensional rod of infinite length, which has an initial temperature at location x given by $g(x)$, for $x \in \mathbb{R}$. If the initial temperature along the rod is uneven, the heat will flow through the rod as time goes on, tending toward equilibrium. Thus, let the function $u(x,t)$ denote the temperature of the rod at location x at time $t \geq 0$.

In this sense, $u(x,0) = g(x)$ correspond to the "initial conditions" for our problem. The formula governing heat flow is given in differential form:

$$\frac{\partial}{\partial t}u(x,t) = \alpha \frac{\partial^2}{\partial x^2}u(x,t),$$

where $\alpha > 0$ is a given positive constant defined by the thermal diffusivity of the material composing the rod. We will solve this equation using Fourier transforms.

a) To start, fix t, so that $u(x,t)$ is viewed as just a function of x. We'll take the Fourier transform of $u(x,t)$, where x is the variable. That is, you should view the Fourier transform as a map of the form

$$\mathcal{F} : u(x,t) \mapsto U(j\omega, t).$$

Show that

$$\frac{\partial}{\partial t}U(j\omega,t) = -\alpha\omega^2 U(j\omega,t).$$

b) Next, fix ω, and think of $U(j\omega,t)$ as just being a function of t. Show that

$$U(j\omega,t) = G(j\omega)e^{-\alpha\omega^2 t},$$

where $g(x) \longleftrightarrow G(j\omega)$.

c) For given constant $\kappa > 0$, derive the transform pair:

$$\frac{1}{2\sqrt{\pi\kappa}}e^{-x^2/4\kappa} \longleftrightarrow e^{-\kappa\omega^2},$$

where x is the 'time domain' variable.

[This is independent of the parts (a) and (b). There are several ways to prove this.]

d) Use parts (b) and (c) to find an explicit expression for $u(x,t)$ in terms of a convolution integral.

e) Assuming $g(x) = \delta(x)$, sketch plots of $u(x,t)$ for $t = 1/(2\alpha)$ and $t = 3/(2\alpha)$ (on the same plot). Your illustration does not need to be exact, just focus on sketching the general shape with the correct height.

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