EE 16B Midterm 2, March 21, 2017

Name: Solutions for HW

SID #: ________________

Discussion Section and TA: ________________

Lab Section and TA: ________________

Name of left neighbor: ________________

Name of right neighbor: ________________

Important Instructions:

- **Show your work.** An answer without explanation is not acceptable and does not guarantee any credit.
- **Only the front pages will be scanned and graded.** You can use the back pages as scratch paper.
- **Do not remove pages**, as this disrupts the scanning. Instead, cross the parts that you don't want us to grade.

Problem	**Points**
1	10
2	15
3	10
4	20
5	15
6	15
7	15
Total	100

1. (10 points) The thirteenth century Italian mathematician Fibonacci described the growth of a rabbit population by the recurrence relation:

$$y(t+2) = y(t+1) + y(t)$$

where $y(t)$ denotes the number of rabbits at month t. A sequence generated by this relation from initial values $y(0)$, $y(1)$ is known as a Fibonacci sequence.

a) (5 points) Bring the recurrence relation above to the state space form using the variables $x_1(t) = y(t)$ and $x_2(t) = y(t+1)$.

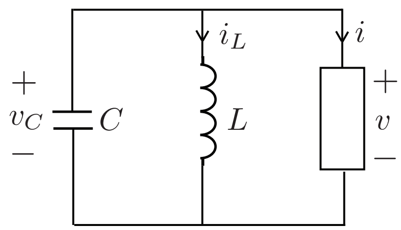
$$\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

b) (5 points) Determine the stability of this system.

Unstable

2. (15 points) Consider the circuit below that consists of a capacitor, an inductor, and a third element with the nonlinear voltage-current characteristic:

$$i = -v + v^3.$$



a) (5 points) Write a state space model of the form

$$\frac{dx_1(t)}{dt} = f_1(x_1(t), x_2(t))$$
$$\frac{dx_2(t)}{dt} = f_2(x_1(t), x_2(t))$$

using the states $x_1(t) = v_C(t)$ and $x_2(t) = i_L(t)$.

$f_1(x_1, x_2) = \frac{1}{C}(x_1 - x_1^3 - x_2)$

$f_2(x_1, x_2) = \frac{1}{L} x_1$

b) (5 points) Linearize the state model at the equilibrium $x_1 = x_2 = 0$ and specify the resulting A matrix.

$$A = \begin{bmatrix} \frac{1}{C} & -\frac{1}{C} \\ \frac{1}{L} & 0 \end{bmatrix}$$

c) (5 points) Determine stability based on the linearization.

Unstable

3. (10 points) Consider the discrete-time system

$$\vec{x}(t+1) = A\vec{x}(t) + Bu(t)$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

a) (5 points) Determine if the system is controllable.

Not controllable

b) (5 points) Explain whether or not it is possible to move the state vector from $\vec{x}(0) = 0$ to

$$\vec{x}(T) = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}.$$

If your answer is yes, specify the smallest possible time T and an input sequence $u(0), \ldots, u(T-1)$ to accomplish this task.

yes $T = 2$

$u(0) = 2$

$u(1) = 1$

4. (20 points) Consider the system

$$\vec{x}(t+1) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

where θ is a constant.

a) (5 points) For which values of θ is the system controllable?

Controllable for all

$\theta \neq K\pi$

where K is an integer

b) (10 points) Select the coefficients k_1, k_2 of the state feedback controller

$$u(t) = k_1 x_1(t) + k_2 x_2(t)$$

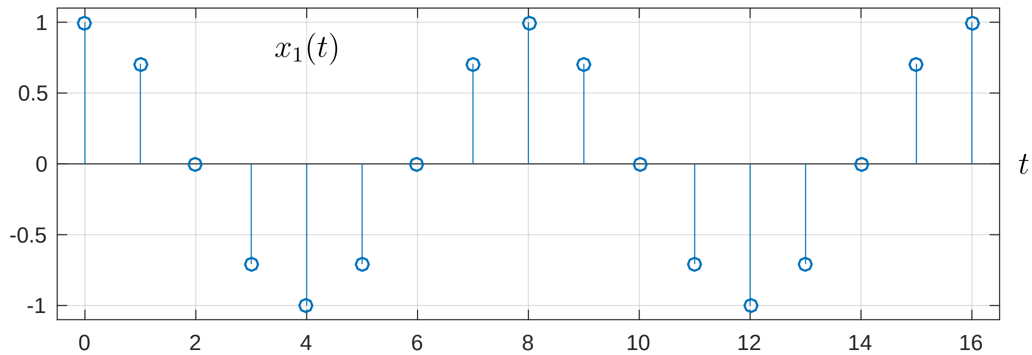
such that the closed-loop eigenvalues are $\lambda_1 = \lambda_2 = 0$. Your answer should be symbolic and well-defined for the values of θ you specified in part (a).

$$K_1 = \frac{2\cos^2\theta - 1}{\sin\theta} = \frac{\cos^2\theta - \sin^2\theta}{\sin\theta}$$

$$K_2 = -2\cos\theta$$

Additional workspace for Problem 4b.

c) (5 points) Suppose the state variable $x_1(t)$ evolves as depicted below when no control is applied ($u = 0$). What is the value of θ?

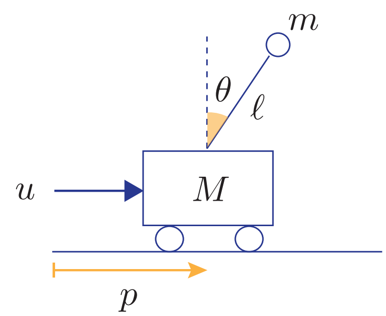


$\theta = \frac{\pi}{4}$

OR

$\theta = -\frac{\pi}{4}$

5. (15 points) Consider the inverted pendulum below, where $p(t)$ is the position of the cart, $\theta(t)$ is the angle of the pendulum, and $u(t)$ is the input force.



When linearized about the upright position, the equations of motion are

$$\begin{aligned} \ddot{p}(t) &= -\frac{m}{M} g\,\theta(t) + \frac{1}{M} u(t) \\ \ddot{\theta}(t) &= \frac{M+m}{M\ell} g\,\theta(t) - \frac{1}{M\ell} u(t) \end{aligned} \tag{1}$$

where M, m, ℓ, g are positive constants.

a) (5 points) Using (1) write the state model for the vector

$$\vec{x}(t) = \begin{bmatrix} p(t) & \dot{p}(t) & \theta(t) & \dot{\theta}(t) \end{bmatrix}^T.$$

$$\frac{d}{dt}\begin{bmatrix} p(t) \\ \dot{p}(t) \\ \theta(t) \\ \dot{\theta}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{m}{M}g & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{M+m}{M\ell}g & 0 \end{bmatrix} \begin{bmatrix} p(t) \\ \dot{p}(t) \\ \theta(t) \\ \dot{\theta}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ \frac{-1}{M\ell} \end{bmatrix} u(t)$$

b) (5 points) Suppose we measure only the position; that is, the output is $y(t) = x_1(t)$. Determine if the system is observable with this output.

Observable

c) (5 points) Suppose we measure only the angle; that is, the output is $y(t) = x_3(t)$. Determine if the system is observable with this output.

unobservable

6. (15 points) Consider the system

$$\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \\ x_3(t+1) \end{bmatrix} = \underbrace{\begin{bmatrix} 0.9 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}, \quad y(t) = \underbrace{\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}}_{C} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}.$$

a) (5 points) Select values for ℓ_1, ℓ_2, ℓ_3 in the observer below such that $\hat{x}_1(t)$, $\hat{x}_2(t)$, $\hat{x}_3(t)$ converge to the true state variables $\vec{x}_1(t)$, $\vec{x}_2(t)$, $\vec{x}_3(t)$ respectively.

$$\begin{bmatrix} \hat{x}_1(t+1) \\ \hat{x}_2(t+1) \\ \hat{x}_3(t+1) \end{bmatrix} = \begin{bmatrix} 0.9 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \\ \hat{x}_3(t) \end{bmatrix} + \underbrace{\begin{bmatrix} \ell_1 \\ \ell_2 \\ \ell_3 \end{bmatrix}}_{L} (\hat{x}_2(t) - y(t)).$$

Let $\lambda_1, \lambda_2, \lambda_3$ be the eigenvalues of $A+LC$

ℓ_1 is arbitrary

$\ell_2 = \lambda_2 + \lambda_3 - 1$

$\ell_3 = -\lambda_2 \lambda_3 - 1$

λ_2, λ_3 must be inside unit circle

Additional workspace for Problem 6a.

b) (5 points) Professor Arcak found a solution to part (a) that guarantees convergence of $\hat{x}_3(t)$ to $x_3(t)$ in one time step; that is

$$\hat{x}_3(t) = x_3(t) \quad t = 1, 2, 3, \ldots$$

for any initial $\vec{x}(0)$ and $\hat{x}(0)$. Determine his ℓ_3 value based on this behavior of the observer. Explain your reasoning.

$\ell_3 = -1$

c) (5 points) When Professor Arcak solved part (a), he found the convergence of $\hat{x}_1(t)$ to $x_1(t)$ to be rather slow no matter what L he chose. Explain the reason why no choice of L can change the convergence rate of $\hat{x}_1(t)$ to $x_1(t)$.

Hint

the system is not observable

7. (15 points) Consider a system with the symmetric form

$$\frac{d}{dt}\begin{bmatrix}\vec{x}_1(t)\\ \vec{x}_2(t)\end{bmatrix} = \begin{bmatrix} F & H \\ H & F \end{bmatrix}\begin{bmatrix}\vec{x}_1(t)\\ \vec{x}_2(t)\end{bmatrix} + \begin{bmatrix} G \\ G \end{bmatrix}\vec{u}(t), \tag{2}$$

where $\vec{x}_1$ and $\vec{x}_2$ have identical dimensions and, therefore, F and H are square matrices.

a) (5 points) Define the new variables

$$\vec{z}_1 = \vec{x}_1 + \vec{x}_2 \quad \text{and} \quad \vec{z}_2 = \vec{x}_1 - \vec{x}_2,$$

and write a state model with respect to these variables:

$$\frac{d}{dt}\begin{bmatrix}\vec{z}_1(t)\\ \vec{z}_2(t)\end{bmatrix} = \left[\begin{array}{c|c} F+H & 0 \\ \hline 0 & F-H \end{array}\right]\begin{bmatrix}\vec{z}_1(t)\\ \vec{z}_2(t)\end{bmatrix} + \left[\begin{array}{c} 2G \\ \hline 0 \end{array}\right] u(t).$$

b) (5 points) Show that the system (2) is not controllable.

Hint 1

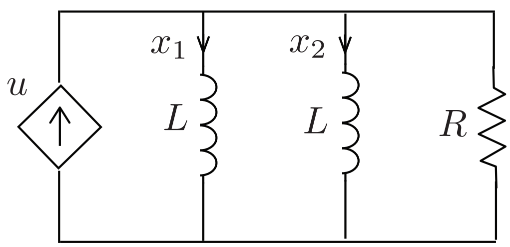
How does $u(t)$ affect $z_1(t)$ and $z_2(t)$?

(if at all)

Hint 2

Check rank conditions

c) (5 points) Write a state model for the circuit below using the inductor currents as the variables. Show that the model has the symmetric form (2).



$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \overset{=F}{-\frac{R}{L}} & \overset{=H}{-\frac{R}{L}} \\ \underset{=H}{-\frac{R}{L}} & \underset{=F}{-\frac{R}{L}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \overset{=G}{\frac{R}{L}} \\ \underset{=G}{\frac{R}{L}} \end{bmatrix} u$$