**This is a closed book, closed notes exam. You need to justify every one of your answers unless you are asked not to do so. Completely correct answers given without justification will receive little credit. Look over the whole exam to find problems that you can do quickly. Hand in this exam before you leave.**

**80 min. 75 points in total. The raw score will be normalized according to the course policy to count into the final score.**

DO NOT tear out any page or add any page. This is crucial for the grading process with gradescope. Write your name on the top left corner of each page. If your answer appears in the scratch paper appended in the end, refer to your answer using the page number.

Your name: ______

Your SID : ______

1. (15 points. This problem contains 3 questions.)
Let

$$\vec{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \vec{a}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \vec{a}_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \vec{b} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

Answer the following questions

(a) Do the columns of $A$ span $\mathbb{R}^3$? Justify your answer.

(b) Are the columns of $A$ linearly independent? Justify your answer.

(c) Represent $\vec{b}$ as the linear combination of $\vec{a}_1, \vec{a}_2, \vec{a}_3$. Is this representation unique? Justify your answer.

2. (25 points. This problem contains 5 questions.)

True or False: If True, explain why. If False, give an explicit numerical example for which the statement does not hold.

(a) $A \in \mathbb{R}^{n \times n}$ is invertible, then $A^{-1}$ is invertible.

(b) If $n$ vectors in $\mathbb{R}^m$ are linearly dependent, then any vector can be represented by the linear combination of other $n-1$ vectors ($n > 1$).

(c) $A \in \mathbb{R}^{n \times n}$, then $(A^T)^2 = (A^2)^T$.

(d) Every subspace of $\mathbb{R}^n$ contains at most $n$ vectors.

(e) Let $\vec{v}_1, \vec{v}_2, \vec{v}_3 \in \mathbb{R}^n$. $\vec{v}_1$ and $\vec{v}_2$ are linearly dependent, then $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly dependent.

3. (10 points. This problem contains 2 questions.)
a) Compute $C = A^T B$, where

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

b) Compute the matrix inverse of

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

4. (25 points. This problem contains 5 questions.)
A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ has the following effect

$$T\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

(a) Compute

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right), T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right).$$

(b) Write down the standard matrix of $T$, denoted by $A$.

(c) Find a basis for null space and column space of $A$.

(d) Is $T$ injective? Is $T$ surjective? Justify your answer.

(e) State the rank theorem, and verify the rank theorem for $A$ from the computation in (c).