This is a closed book, closed notes exam. You need to justify every one of your answers unless you are asked not to do so. Completely correct answers given without justification will receive little credit. Look over the whole exam to find problems that you can do quickly. Hand in this exam before you leave.

80 min. 65 points in total. The raw score will be normalized according to the course policy to count into the final score.

DO NOT tear out any page or add any page. This is crucial for the grading process with gradescope. Write your name on the top left corner of each page. If your answer appears in the scratch paper appended in the end, refer to your answer using the page number.

Your name: ____

Your SID : ____

1. (10 points) $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{bmatrix}$.

Find all its eigenvalues and the corresponding eigenvectors.

A:

(2 points)

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & -1 & 0 \\ -1 & 3-\lambda & -1 \\ 0 & -1 & 2-\lambda \end{vmatrix}.$$

(2 points) Evaluate this we have

$$|A - \lambda I| = (2-\lambda)(1-\lambda)(4-\lambda)$$

So the eigenvalues are $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 4$.

(2 points) For λ_1, solve

$$(A - I)\vec{v}_1 = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \vec{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

gives

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

(2 points) Similarly solve $(A - 2I)\vec{v}_2 = 0$ gives

$$\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

(2 points) Solve $(A - 4I)\vec{v}_3 = 0$ gives

$$\vec{v}_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

2. True or False (15 points) If True, explain why. If False, give a counterexample. The correct answer is worth 1 point for each problem. The rest of the points come from the justification.

(a) (3 points) If $A \in \mathbb{R}^{n\times n}$ with $\det(A) = 3$, then $\det(A^{-1}) = \frac{1}{3}$.

A: True. Because $\det(A)\det(A^{-1}) = \det(I) = 1$, we have

$$3 \cdot \det(A^{-1}) = 1,$$

and hence $\det(A^{-1}) = \frac{1}{3}$.

(b) (3 points) If $A \in \mathbb{R}^{n\times n}$ is an invertible matrix, and A is diagonalizable, then A^{-1} is diagonalizable.

A: True. A is diagonalizable then $A = PDP^{-1}$, where P is an invertible matrix, and D is a diagonal matrix. Since A is invertible, all diagonal entries of D are not zero. Hence

$$A^{-1} = (PDP^{-1})^{-1} = PD^{-1}P^{-1},$$

and A^{-1} is diagonalizable.

(c) (3 points) The vector spaces $\mathbb{P}_3$ and $\mathbb{R}^3$ are isomorphic.

A: False. Since $\dim\mathbb{P}_3 = 4$, and $\dim\mathbb{R}^3 = 3$, they cannot be isomorphic.

(d) (3 points) V is a vector space. If $\dim V = n$ and S is a linearly independent set in V, then S is a basis for V.

A: False. For instance $V = \mathbb{R}^2$, and $S = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$. S is a linearly independent set but is not a basis for V.

(e) (3 points) If $A, B \in \mathbb{R}^{2\times 2}$ and both A and B are diagonalizable, then AB is diagonalizable.

A: False. Counter example:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}.$$

Both A, B are diagonalizable, but

$$AB = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

is not diagonalizable.

3. (10 points) Let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\} \equiv \{1, t-1, (t-1)^2\}$ be a subset of $\mathbb{P}_2$.

(a) (5 points) Show that $\mathcal{B}$ is a basis for $\mathbb{P}_2$.

A:

(3 points) The coordinates of $\{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ in the standard basis $\{1, t, t^2\}$ form the following column matrix

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}.$$

(2 points) This is already in row echelon form. It has a pivot in each row and each column. Hence the set $\mathcal{B}$ is linearly independent, and span$\mathcal{B} = \mathbb{P}_2$. So $\mathcal{B}$ is a basis for $\mathbb{P}_2$.

(b) (5 points) Find the $\mathcal{B}$-coordinate of $1 + 2t + 3t^2$.

(2 points) We need to find

$$1 + 2t + 3t^2 = a_1 + a_2(t-1) + a_3(t-1)^2.$$

(2 points) Match the coefficient on both sides we have

$$1 + 2t + 3t^2 = 6 + 8(t-1) + 3(t-1)^2,$$

(1 points) the $\mathcal{B}$-coordinate is

$$\begin{bmatrix} 6 \\ 8 \\ 3 \end{bmatrix}.$$

4. (10 points) Let $\mathcal{B} = \left\{ \begin{bmatrix} 7 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\}$, $\mathcal{C} = \left\{ \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \end{bmatrix} \right\}$.

(a) (7 points) Find the change-of-coordinates matrix $\underset{C\leftarrow B}{P}$.

A: (2 points) Recall that you may find $\underset{C\leftarrow B}{P}$ by using the fact that

$$\underset{C\leftarrow B}{P} = \underset{C\leftarrow\mathcal{E}}{P}\,\underset{\mathcal{E}\leftarrow B}{P} = \left(\underset{\mathcal{E}\leftarrow C}{P}\right)^{-1}\underset{\mathcal{E}\leftarrow B}{P}.$$

(2 points) Since

$$\underset{\mathcal{E}\leftarrow C}{P} = \begin{bmatrix} 4 & 5 \\ 1 & 2 \end{bmatrix}, \quad \underset{\mathcal{E}\leftarrow B}{P} = \begin{bmatrix} 7 & 2 \\ -2 & -1 \end{bmatrix},$$

(3 points) Then we have

$$\underset{C\leftarrow B}{P} = \begin{bmatrix} 8 & 3 \\ -5 & -2 \end{bmatrix}.$$

(b) (3 points) Find $[x]_{\mathcal{C}}$, where $[x]_{\mathcal{B}} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$.

$$[\vec{x}]_C = \underset{C\leftarrow B}{P}[\vec{x}]_B = \begin{bmatrix} 47 \\ -30 \end{bmatrix}.$$

5. (10 points) Consider the subspace W of $\mathbb{R}^4$ spanned by

$$\vec{u} = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 2 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 4 \end{bmatrix}$$

(a) (5 points) Find a nonzero vector $\vec{w}$ in W that is orthogonal to $\vec{u}$.

A:

(3 points) Let $\vec{w} = a\vec{u} + b\vec{v}$, and we would like to have $\vec{w} \cdot \vec{u} = 0$.

(2 points) Since

$$\vec{w} \cdot \vec{u} = 9a + 9b,$$

one possibility is $a = 1, b = -1$, and

$$\vec{w} = \vec{u} - \vec{v} = \begin{bmatrix} 0 \\ 1 \\ -2 \\ -2 \end{bmatrix}.$$

(b) (5 points) Find the orthogonal projection of the vector

$$\vec{y} = \begin{bmatrix} 3 \\ -1 \\ 2 \\ 1 \end{bmatrix}$$

to the subspace W.

A:

(3 points) The projection is

$$\text{proj}_W \vec{y} = \frac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \vec{u} + \frac{\vec{y} \cdot \vec{w}}{\vec{w} \cdot \vec{w}} \vec{w}$$

(2 points) Evaluate this, and we have

$$\text{proj}_W \vec{y} = \frac{1}{9} \vec{u} - \frac{7}{9} \vec{w} = \frac{1}{9} \begin{bmatrix} 1 \\ -7 \\ 12 \\ 16 \end{bmatrix}.$$

6. (10 points)

Let $T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ be the linear transformation given by $T(A) = A^T$, where A^T is the transpose of A.

(a) (2 points) Is T an isomorphism? If so, write down T^{-1}.
A: Yes. Since $(A^T)^T = A$, we have $T^{-1} = T$.
(b) (2 points) Find $[T]_{\mathcal{B}}$, which is the matrix representation of T under the basis

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \right\}$$

A:

$$[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(c) (6 points) Find the eigenvalues and the eigenspaces of $[T]_{\mathcal{B}}$.
A:
(2 points) The characteristic equation is

$$(1-\lambda)^3(1+\lambda) = 0.$$

Hence the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = -1$.
(2 points) For $\lambda_1 = 1$, the eigenspace of $[T]_{\mathcal{B}}$ is

$$\mathrm{Null}([T]_{\mathcal{B}} - I) = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

(2 points) For $\lambda_2 = -1$, the eigenspace is

$$\mathrm{Null}(A + I) = \mathrm{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \right\}.$$

Note: if you write in the form

$$\mathrm{span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

and

$$\mathrm{span}\left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$$

you also receive the full credit due to the slight ambiguity at the end of the exam!