DO NOT TURN OVER UNTIL INSTRUCTED TO DO SO.

Formulae

$$\int \tan(x)\, dx = \ln|\sec(x)| + C \qquad \int \sec(x)\, dx = \ln|\sec(x) + \tan(x)| + C$$

$$\int \frac{1}{1+x^2} dx = \arctan(x) + C \qquad \int \frac{1}{\sqrt{1-x^2}} dx = \arcsin(x) + C$$

$$\frac{d\tan(x)}{dx} = \sec^2(x) \qquad \frac{d\sec(x)}{dx} = \tan(x)\sec(x)$$

$$1 = \sin^2(x) + \cos^2(x) \qquad 1 + \tan^2(x) = \sec^2(x)$$

$$\cos^2(x) = \frac{1+\cos(2x)}{2} \qquad \sin^2(x) = \frac{1-\cos(2x)}{2}$$

$$e^x = 1 + x + \tfrac{x^2}{2} + \tfrac{x^3}{6} + \cdots = \textstyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\sin x = x - \tfrac{x^3}{3!} + \tfrac{x^5}{5!} - \tfrac{x^7}{7!} + \tfrac{x^9}{9!} - \cdots = \textstyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\cos x = 1 - \tfrac{x^2}{2!} + \tfrac{x^4}{4!} - \tfrac{x^6}{6!} + \tfrac{x^8}{8!} - \cdots = \textstyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$\arctan x = x - \tfrac{x^3}{3} + \tfrac{x^5}{5} - \tfrac{x^7}{7} + \tfrac{x^9}{9} - \cdots = \textstyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

$$\ln(1+x) = x - \tfrac{x^2}{2} + \tfrac{x^3}{3} - \tfrac{x^4}{4} + \tfrac{x^5}{5} - \cdots = \textstyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$$

$$\lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n = e$$

CALCULATORS ARE NOT PERMITTED

Name and section: 1B Midterm 2 (002) Master

GSI's name: ____________________

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This exam consists of 5 questions. Answer the questions in the spaces provided.

1. Determine if the following series converge or diverge. If convergent you do not need to give the sum. Carefully justify your answers.

   (a) (10 points)

$$\sum_{n=1}^{\infty} \sin\left(\frac{\pi n}{2n+1}\right)$$

Solution:

$$\lim_{n\to\infty} \frac{\pi n}{2n+1} = \frac{\pi}{2} \Rightarrow \lim_{n\to\infty} \sin\left(\frac{\pi n}{2n+1}\right) = 1 \neq 0$$

$\Rightarrow$ Divergent by divergence test.

   (b) (15 points)

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+1}$$

Solution:

$$a_n = \frac{\sqrt{n}}{n+1}, \quad b_n = \frac{1}{\sqrt{n}} \Rightarrow \frac{a_n}{b_n} = \frac{n}{n+1}$$

Hence $\frac{a_n}{b_n} \to 1$ as $n \to \infty$

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \text{ div } \underset{\substack{\uparrow \\ \text{Limit comparison}}}{\Rightarrow} \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+1} \text{ div.}$$

$(p = \frac{1}{2} < 1)$

2. (20 points) Determine if the following series is convergent or divergent. If convergent you do not need to determine the sum.

$$\sum_{n=1}^{\infty} \frac{1 \cdot 4 \cdot 7 \cdots (3n-2)}{(2n)!}$$

Solution:

$$a_n = \frac{1 \cdot 4 \cdot 7 \ldots (3n-2)}{(2n)!}$$

$$\Rightarrow \left|\frac{a_{n+1}}{a_n}\right| = \frac{3n+1}{(2n+1)(2n+2)} \longrightarrow 0 \quad \text{as } n \to \infty$$

$\Rightarrow$ Convergent by ratio test.

3. (30 points) Using the integral test, determine whether the following series is absolutely convergent, conditionally convergent or divergent. If convergent you do not need to determine the sum.

$$\sum_{n=1}^{\infty}(-1)^n\frac{n}{e^n}$$

Solution:

$f(x) = x e^{-x}$

1/ $f(x) \geq 0$ on $[1, \infty)$

2/ $f'(x) = e^{-x} - x^2 e^{-x} = (1-x^2)e^{-x} \leq 0$ on $[1,\infty) \Rightarrow f(x)$ decreasing.

3/ $e^{x} \neq 0 \Rightarrow f(x)$ Continuous on $[1,\infty)$

By Parts

$$\int x e^{-x}\,dx = -xe^{-x} + \int e^{-x}\,dx = -xe^{-x} - e^{-x} + C$$

$$\int_1^{\infty} xe^{-x}\,dx = \lim_{t\to\infty}\left(\frac{-t}{e^t} - \frac{1}{e^t}\right) - \left(-2e^{-1}\right) = 2e^{-1}$$

$$\left(\lim_{t\to\infty}\frac{t}{e^t} = \lim_{t\to\infty}\frac{1}{e^t} = 0\right)$$

L'Hospital

$$\int_1^{\infty} f(x)\,dx \text{ conv} \Rightarrow \sum_{n=1}^{\infty}\frac{n}{e^n} \text{ conv} \Rightarrow \text{Absolutely converged}$$

Integral Test

4. (25 points) Determine the domain of the function $f(x)$ given by the power series

$$\sum_{n=1}^{\infty}(-1)^n\frac{(2x-4)^n}{n^{3/2}}.$$

Solution:

$$a_n = (-1)^n \frac{(2x-4)^n}{n^{3/2}} \Rightarrow \left|\frac{a_{n+1}}{a_n}\right| = \left(\frac{n}{n+1}\right)^{3/2}|2x-4|$$

$$\Rightarrow \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |2x-4|$$

$|2x-4| < 1$ (ie $|x-2| < \frac{1}{2}$) $\Rightarrow$ conv.

$|2x-4| > 1$ (ie $|x-2| > \frac{1}{2}$) $\Rightarrow$ div

$\Rightarrow R = \frac{1}{2}$

$x = 3/2$, $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$ conv. ($p = 3/2 > 1$)

$x = 5/2$, $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n^{3/2}}$ conv as absolutely conv.

$\Rightarrow$ Domain $= [3/2, 5/2]$.

5. (a) (15 points) Calculate the Taylor series at $a = 1$ of the function

$$f(x) = (1-x)^2 \arctan(1-x).$$

Make sure to write a general term of the series. Hint: First consider Maclaurin series of $\arctan(x)$.

Solution:

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} \cdots \qquad |x| < 1$$

$$\Rightarrow \arctan(1-x) = (1-x) - \frac{(1-x)^3}{3} + \frac{(1-x)^5}{5} \cdots \qquad |1-x| < 1$$

$$\Rightarrow (1-x)^2 \arctan(1-x)$$

$$= (1-x)^3 - \frac{(1-x)^5}{3} + \frac{(1-x)^7}{5} \cdots \qquad |x-1| < 1$$

$$= -(x-1)^3 + \frac{(x-1)^5}{3} - \frac{(x-1)^7}{5} \cdots$$

$$= \sum_{n=1}^{\infty} (-1)^n \frac{(x-1)^{2n+1}}{2n-1} \leftarrow \text{Taylor Series at } a = 1.$$

(b) (10 points) Using part (a), or otherwise, calculate $f^{(101)}(1)$. You do not need to simplify your answer.

Solution:

$$\frac{f^{(101)}(1)}{101!} = \frac{(-1)^{50}}{2\cdot 50 - 1} = \frac{1}{99} \Rightarrow f^{(101)}(1) = \frac{(101)!}{99}.$$

END OF EXAM

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