Math 1B. Solutions to the Second Midterm

1. (16 points) Find the first four terms of the Maclauren series for

$$f(x) = \frac{\cos x}{1 + \ln(1+x)}\ .$$

Note that you may want to find this in a manner other than by direct differentiation of the function.

From the formulas on the front of the exam, we have

$$\cos x = 1 - \frac{x^2}{2} + \dots \quad \text{and}$$
$$1 + \ln(1+x) = 1 + x - \frac{x^2}{2} + \frac{x^3}{3} - \dots\ .$$

The first four terms go up to the x^3 term. Since both of the above series have nonzero constant term, we can find the answer, valid up to the x^3 term, by long division ignoring all terms past the x^3 term:

$$
\begin{array}{r|rrrr}
 & 1 & -x & +x^2 & -\frac{11}{6}x^3 \\ \cline{2-5}
1+x-\frac{x^2}{2}+\frac{x^3}{3} & 1 & +0x & -\frac{x^2}{2} & +0x^3 \\
 & 1 & +x & -\frac{x^2}{2} & +\frac{x^3}{3} \\ \cline{2-5}
 & & -x & +0x^2 & -\frac{1}{3}x^3 \\
 & & -x & -x^2 & +\frac{1}{2}x^3 \\ \cline{3-5}
 & & & x^2 & -\frac{5}{6}x^3 \\
 & & & x^2 & +x^3 \\ \cline{4-5}
 & & & & -\frac{11}{6}x^3 \\
 & & & & -\frac{11}{6}x^3 \\ \cline{5-5}
 & & & & 0
\end{array}
$$

Thus

$$\frac{\cos x}{1+\ln(1+x)} = 1 - x + x^2 - \frac{11}{6}x^3 + \dots\ .$$

2. (18 points) (a). Find $T_2(x)$, the degree 2 Taylor polynomial of the function $f(x) = \sqrt{x}$ at $a = 100$.

We have

$$
\begin{aligned}
f(x) &= \sqrt{x} & f(100) &= 10 \\
f'(x) &= \tfrac{1}{2}x^{-1/2} & f'(100) &= \tfrac{1}{2}\cdot\tfrac{1}{10} = \tfrac{1}{20} \\
f''(x) &= -\tfrac{1}{4}x^{-3/2} & f''(100) &= -\tfrac{1}{4}\cdot\tfrac{1}{10^3} = -\tfrac{1}{4000}\ ,
\end{aligned}
$$

and therefore

$$T_2(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2$$
$$= 10 + \frac{1}{20}(x-100) - \frac{1}{8000}(x-100)^2 \ .$$

(b). How accurate is the approximation $T_2(x) \approx \sqrt{x}$ when $99.9 \le x \le 100.1$?

We use Taylor's Inequality with $n = 2$ and $d = 0.1$. We have

$$f^{(n+1)}(x) = f'''(x) = \frac{3}{8}x^{-5/2} \ .$$

On the interval $[99.9, 100.1]$ this has its maximum absolute value at $x = 99.9$, so we can take

$$M = \frac{3}{8 \cdot 99.9^{5/2}} \ .$$

Therefore,

$$|f(x) - T_2(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1}$$
$$\le \frac{3(0.1)^3}{8(3!)(99.9)^{5/2}} \ .$$

3. (18 points) Find the partial fraction decomposition of

$$\frac{x^3+2x}{x^3+1} = \frac{x^3+2x}{(x+1)(x^2-x+1)} \ .$$

First, note that the fraction is not a proper fraction:

$$\frac{x^3+2x}{x^3+1} = 1 + \frac{2x-1}{x^3+1} \ .$$

Given that the denominator x^3+1 factors as $(x+1)(x^2-x+1)$ (and that the quadratic factor has no real roots), the form of the partial fraction decomposition is as below.

Clearing denominators in the equation

$$\frac{2x-1}{x^3+1} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}$$

gives

$$2x-1 = A(x^2-x+1) + (Bx+C)(x+1) \ .$$

Next, we plug some values into this equality in order to get equations in the unknowns A, B, and C:

$$\begin{aligned} x &= -1 &\implies&& 3A &= -3 \\ x &= 0 &\implies&& A + C &= -1 \\ x &= 1 &\implies&& A + 2B + 2C &= 1\ . \end{aligned}$$

The first equation gives $A = -1$; using this, the second equation gives $C = 0$; finally, using these two values, the third equation gives $B = 1$. Therefore, the partial fraction decomposition is

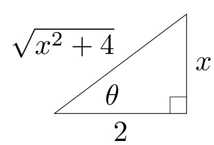
$$\frac{x^3 + 2x}{x^3 + 1} = 1 - \frac{1}{x+1} + \frac{x}{x^2 - x + 1}\ .$$

4. (18 points) Find $\displaystyle\int \frac{dx}{x^2\sqrt{x^2+4}}$.

Substitute $x = 2\tan\theta$. Then $dx = 2\sec^2\theta\, d\theta$ and

$$\sqrt{x^2+4} = \sqrt{4\tan^2\theta + 4} = 2\sqrt{\tan^2\theta + 1} = 2\sec\theta\ .$$

You can also see the latter by drawing a right triangle:



The integral is then

$$\begin{aligned} \int \frac{dx}{x^2\sqrt{x^2+4}} &= \int \frac{2\sec^2\theta\, d\theta}{4\tan^2\theta \cdot 2\sec\theta} \\ &= \frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta}\, d\theta \\ &= \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\, d\theta \\ &= \frac{1}{4}\int \csc\theta\cot\theta\, d\theta \\ &= -\frac{1}{4}\csc\theta + C \\ &= -\frac{1}{4}\cdot\frac{\sqrt{x^2+4}}{x} + C \\ &= -\frac{\sqrt{x^2+4}}{4x} + C\ . \end{aligned}$$

(Here we used the triangle to get the next to last line.)

5. (15 points) (a). Find the arc length of the curve $y = \dfrac{x^2}{2} - \dfrac{\ln x}{4}$, $1 \le x \le 3$.

First, we have

$$y' = x - \frac{1}{4x} ,$$

so

$$1 + (y')^2 = 1 + \left(x - \frac{1}{4x}\right)^2 = 1 + x^2 - \frac{1}{2} + \frac{1}{16x^2} = x^2 + \frac{1}{2} + \frac{1}{16x^2} = \left(x + \frac{1}{4x}\right)^2 ,$$

so the arc length is

$$\int_1^3 \sqrt{1 + (y')^2}\, dx = \int_1^3 \left(x + \frac{1}{4x}\right) dx = \left[\frac{x^2}{2} + \frac{\ln x}{4}\right]_1^3 = \frac{9}{2} + \frac{\ln 3}{4} - \frac{1}{2} = 4 + \frac{\ln 3}{4} .$$

(b). Find the arc length function for this curve, with starting point $(1, 1/2)$.

The integrand is the same as in part (a), except that x is changed to t. The arc length function is:

$$s(x) = \int_1^x \sqrt{1 + \left(\frac{d}{dt}\left(\frac{t^2}{2} - \frac{\ln t}{4}\right)\right)^2}\, dt = \left[\frac{t^2}{2} + \frac{\ln t}{4}\right]_1^x = \frac{x^2}{2} + \frac{\ln x}{4} - \frac{1}{2} .$$

6. (15 points) A lamina with uniform density $2\,\mathrm{g/cm^2}$ occupies the region in the xy-plane bounded by the curves $y = x^2$, $y = 9x$, and $x = 3$. Here x and y are measured in cm.

Find the moments of the lamina with respect to the x- and y-axes.

(Actually, there are two regions bounded by the indicated curves:

$$x^2 \le y \le 9x , \quad 0 \le x \le 3 \qquad \text{and} \qquad x^2 \le y \le 9x , \quad 3 \le x \le 9 .$$

We will give the answer for the first region. The answer for the other region is computed by integrating the same integrands from 3 to 9.)

The moment with respect to the y-axis is

$$M_y = 2\int_0^3 x(9x - x^2)\, dx = 2\int_0^3 (9x^2 - x^3)\, dx = 2\left[3x^3 - \frac{x^4}{4}\right]_0^3 = \left(6 \cdot 3^3 - \frac{3^4}{2}\right) \mathrm{g\,cm} ,$$

and the moment with respect to the x-axis is

$$M_x = 2\int_0^3 \frac{1}{2}((9x)^2 - (x^2)^2)\, dx = \int_0^3 (81x^2 - x^4)\, dx$$
$$= \left[27x^3 - \frac{x^5}{5}\right]_0^3 = \left(27 \cdot 3^3 - \frac{3^5}{5}\right) \mathrm{g\,cm} .$$