Math 1B. Solutions to the First Midterm

1. (7 points) Find $\int \sqrt{x} \ln x \, dx$.

Do integration by parts with:

$$\begin{array}{ll} u = \ln x & dv = \sqrt{x}\, dx \\ du = \frac{dx}{x} & v = \frac{2}{3} x^{3/2} \end{array}$$

This gives

$$\begin{aligned} \int \sqrt{x} \ln x \, dx &= \frac{2}{3} x^{3/2} \ln x - \frac{2}{3} \int x^{3/2} \frac{dx}{x} \\ &= \frac{2}{3} x^{3/2} \ln x - \frac{2}{3} \int \sqrt{x} \, dx \\ &= \frac{2}{3} x^{3/2} \ln x - \frac{4}{9} x^{3/2} + C \ . \end{aligned}$$

2. (10 points) For $n \ge 2$ let $a_n = \left(1 + \frac{1}{2^2}\right)\left(1 + \frac{1}{3^2}\right) \cdots \left(1 + \frac{1}{n^2}\right)$.

(a). Given that the sequence $\{a_n\}$ is bounded above, explain why it converges. (You do not need to find the limit.)

The a_n are all positive, hence bounded below (by zero).

This is an increasing sequence, because $a_{n+1} = a_n \left(1 + \frac{1}{(n+1)^2}\right) > a_n$ (since $1 + \frac{1}{(n+1)^2} > 1$).

Therefore, by the Monotonic Sequence Theorem, the sequence converges.

(b). Show that $\{a_n\}$ *is* in fact bounded above. [**Hint:** Look at $\ln a_n$.]

We have

$$\ln a_n = \ln \prod_{k=1}^{n} \left(1 + \frac{1}{k^2}\right) = \sum_{k=1}^{n} \ln\left(1 + \frac{1}{k^2}\right) \ .$$

Therefore, $\ln a_n$ is the n^{th} partial sum of the series $\sum_{n=1}^{\infty} \ln\left(1 + \frac{1}{n^2}\right)$. This series converges by the Limit Comparison Test, comparing it with the convergent p-series $\sum 1/n^2$:

$$\lim_{n\to\infty} \frac{\ln\left(1 + \frac{1}{n^2}\right)}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{\frac{-\frac{2}{n^3}}{1+\frac{1}{n^2}}}{-\frac{2}{n^3}} = \lim_{n\to\infty} \frac{1}{1 + \frac{1}{n^2}} = 1 \ .$$

If we let $L = \sum_{n=1}^{\infty} \ln\left(1 + \frac{1}{n^2}\right)$, then $\ln a_n < L$ for all n , because each term in the series $\sum \ln(1 + 1/k^2)$ is positive, so all partial sums are less than the total sum L . This shows that the sequence $\{a_n\}$ is bounded above, by e^L .

3. (6 points) If $\sum a_n$ and $\sum b_n$ are both divergent, is $\sum(a_n+b_n)$ necessarily divergent? (If yes, briefly explain why, mentioning theorems from the book as appropriate. If no, give an example that illustrates why it is not always true.)

No, it could converge. For example, if $a_n = 1$ and $b_n = -1$ for all n, then both series $\sum a_n$ and $\sum b_n$ diverge (by the Test for Divergence), but the series $\sum(a_n + b_n) = \sum(1-1) = \sum 0$ converges.

(This is Exercise 84 on page 713, which was among the assigned written exercises.)

4. (9 points) Determine whether the series $\displaystyle\sum_{n=1}^{\infty}\left(\frac{2-\sin n}{4}\right)^n$ is absolutely convergent, conditionally convergent, or divergent.

The terms are all positive, since $1 \le 2 - \sin n \le 3$ for all n. Also,

$$\left(\frac{2-\sin n}{4}\right)^n \le \left(\frac{3}{4}\right)^n$$

for all n. Since the series $\sum(3/4)^n$ is a convergent geometric series, the original series converges by the Comparison Test.

It converges absolutely, because all of its terms are positive.

5. (9 points) Determine whether the series $\displaystyle\sum_{n=2}^{\infty}\frac{(-1)^n}{n\ln n}$ is absolutely convergent, conditionally convergent, or divergent.

Let

$$b_n = \frac{1}{n\ln n} .$$

The denominator is positive and increasing, so $b_n > 0$ for all n and $\{b_n\}$ is a decreasing sequence. Also, $n\ln n \to \infty$ as $n \to \infty$, so $b_n \to 0$ as $n \to \infty$. Therefore, by the Alternating Series Test, the series converges.

To check for absolute convergence, we use the Integral Test. The function $1/(x\ln x)$ is positive and decreasing for all $x \ge 2$, so we can apply this test. Using the substitution $u = \ln x$, $du = dx/x$, we have:

$$\int \frac{dx}{x\ln x} = \int \frac{du}{u} = \ln|u| + C = \ln(\ln x) + C ,$$

so

$$\int_2^{\infty} \frac{dx}{x\ln x} = \lim_{t\to\infty}\int_2^t \frac{dx}{x\ln x} = \lim_{t\to\infty}\left(\ln(\ln t) - \ln(\ln 2)\right) = \infty .$$

Since this limit diverges, so does the series $\sum b_n$.

Therefore the series in the question converges conditionally.

6. (9 points) Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{n! \cdot n^2}{1 \cdot 3 \cdot \cdots \cdot (2n-1)}$ is absolutely convergent, conditionally convergent, or divergent.

We apply the Ratio Test. Since $2(n+1) - 1 = 2n + 2 - 1 = 2n + 1$,

$$
\begin{aligned}
\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \frac{\frac{(n+1)!(n+1)^2}{1\cdot 3\cdot\cdots\cdot(2n+1)}}{\frac{n!\cdot n^2}{1\cdot 3\cdot\cdots\cdot(2n-1)}} \\
&= \lim_{n\to\infty} \left(\frac{(n+1)!}{n!} \cdot \frac{(n+1)^2}{n^2} \cdot \frac{1\cdot 3\cdot\cdots\cdot(2n-1)}{1\cdot 3\cdot\cdots\cdot(2n+1)} \right) \\
&= \lim_{n\to\infty} \left(\frac{(n!)(n+1)}{n!} \cdot \frac{(n+1)^2}{n^2} \cdot \frac{1\cdot 3\cdot\cdots\cdot(2n-1)}{1\cdot 3\cdot\cdots\cdot(2n-1)(2n+1)} \right) \\
&= \lim_{n\to\infty} \left((n+1) \cdot \frac{(n+1)^2}{n^2} \cdot \frac{1}{2n+1} \right) \\
&= \lim_{n\to\infty} \frac{(n+1)^3}{n^2(2n+1)} \\
&= \lim_{n\to\infty} \frac{(1+1/n)^3}{2+1/n} \\
&= \frac{1}{2} \,.
\end{aligned}
$$

Since this limit is < 1, the series converges absolutely.