EECS 16A Designing Information Devices and Systems I
Spring 2017 Babak Ayazifar, Vladimir Stojanovic Midterm 1

Exam location: 2050 VLSB, Last Name: Surpenant-Zzz

PRINT your student ID: ____

PRINT AND SIGN your name: __________________, __________________ ______________________
(last) (first) (signature)

PRINT your Unix account login: ee16a-_____________

PRINT your discussion section and GSI (the one you attend): ____

Name and SID of the person to your left: ____

Name and SID of the person to your right: ____

Name and SID of the person in front of you: ____

Name and SID of the person behind you: ____

1. What other courses are you taking this term? (1 point)

2. What activity do you really enjoy? Describe how it makes you feel. (1 point)

Do not turn this page until the proctor tells you to do so. You may work on the questions above.

PRINT your name and student ID: ____

3. Mechanical Basis (8 points)

(a) (3 points) Let vectors $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3 \in \mathbb{R}^4$:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 2 \\ 0 \\ 3 \\ 0 \end{bmatrix}$$

Can the set of vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ form a basis for the vector space $\mathbb{R}^4$? Justify your answer.

(b) (5 points) Let $\vec{x} = \begin{bmatrix} 3 \\ 5 \\ 4 \end{bmatrix}$. Given a new set of vectors in $\mathbb{R}^3$:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 2 \\ 7 \\ 1 \end{bmatrix}, \vec{v}_4 = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}, \vec{v}_5 = \begin{bmatrix} 1 \\ 6 \\ 3 \end{bmatrix}$$

Can the set of vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5\}$ be a basis for $\mathbb{R}^3$? If so, express $\vec{x}$ as a linear combination of the basis vectors. Otherwise, choose a new basis using $\vec{v}_1$, $\vec{v}_2$ and any number of additional vectors in the set, then express $\vec{x}$ as a linear combination of the newly constructed basis vectors.

4. Eigenvectors (10 points)

(a) (5 points) Find the eigenvectors and associated eigenvalues of $\mathbf{M}$ in terms of a and b.

$$\mathbf{M} = \begin{bmatrix} 1 & a \\ 0 & b \end{bmatrix}$$

(b) (5 points) Let $\mathbf{A} = \begin{bmatrix} 1 & 4 \\ 0 & 2 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 1 & -2 \\ 0 & 0.5 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$. Evaluate $\mathbf{A}^{203}\mathbf{B}^{199}\vec{v}$.

5. Eigenvalue Proof (10 points)

For two square matrices $\mathbf{A}$ and $\mathbf{B}$, show that $\mathbf{AB}$ has the same eigenvalues as $\mathbf{BA}$.

Hint: Show that if $\boldsymbol{AB}$ or $\boldsymbol{BA}$ has an eigenvalue λ, then the other one has the same eigenvalue.

6. Wall Shadows (15 points)

Oh no, someone decided to build a wall between you and your favorite neighbor! They built a huge complicated 3D structure that you cannot fully see. However, you can observe the shadow cast by this structure.

(a) (5 points) Let's first try to model what the shadow of a 3D object might be. Create a transformation that flattens all components of a vector (vectors in 3D space) onto the xy-plane. That is, find a matrix $\mathbf{A}$ such that :

$$\mathbf{A}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$$

(b) (5 points) Suppose you wanted to observe the height of the wall. Could you observe it just by seeing the shadow, assuming the shadow is created by the transformation from the matrix above?

(c) (5 points) One of your friends comes along and wonders, if you apply the matrix again, whether you will get any new information. Realizing your matrix is really just a projection, you argue that projecting twice will give you no new information. We will show this generally for any two vectors. To prove this to your friend, show the following:

For any vectors $\vec{u}$ and $\vec{v}$, if $\vec{x} = proj_{\vec{u}}\vec{v}$ and $\vec{y} = proj_{\vec{u}}\vec{x}$, then $\vec{x} = \vec{y}$.
Recall that the projection of a vector $\vec{v}$ onto $\vec{u}$, $proj_{\vec{u}}\vec{v} = \frac{\langle \vec{v}, \vec{u} \rangle}{||\vec{u}||^2}\vec{u}$.

7. Structured Illumination (20 points)

In the lab, you acquired images using a single pixel imager and a projector. You did this by successively creating different illumination patterns (masks) and recording the total intensity at the single pixel detector. In this problem, consider a 3×3 illumination grid, where we will use structured light patterns (not just single-pixel masks) to acquire image information.

The 9 pixels, represented by x_i, have values that are either 0 (no light reflected by the object at that pixel location) or 1 (light is completely reflected by object at that pixel location). For example, if you use your sensor to look at the Campanile, pixels x_2, x_5, and x_8 would reflect light from an illumination source, or for Cory Hall, the bottom two rows would reflect light, as shown in Fig. **??**

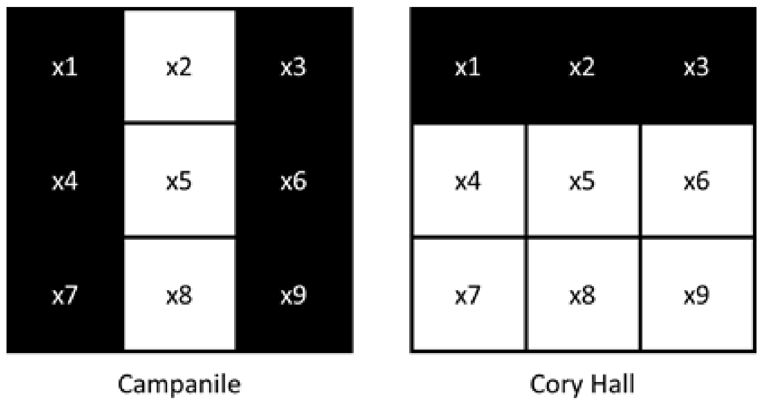


Figure 1: Campus objects represented with 9 pixels

For this problem, we will try to use just two mask patterns to recognize various objects around campus using our sensor.

(a) (2 points) For this part, consider the mask patterns shown in Fig. **??**.

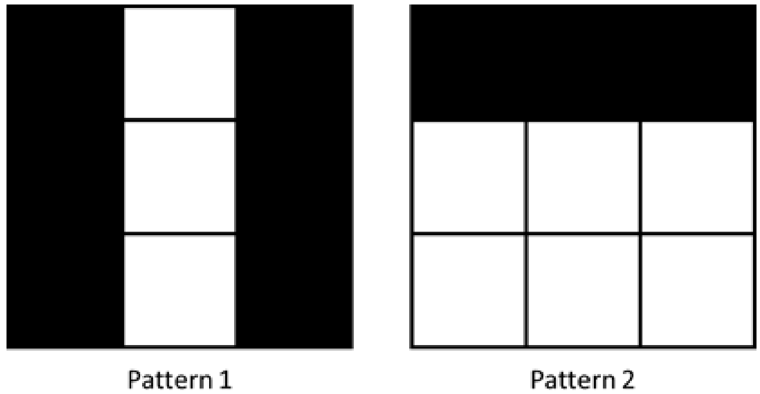


Figure 2: Two scanning patterns

The value detected at our light sensor for a particular mask pattern is the dot-product of the mask pattern with the 9-pixel representation of the object. Using the two mask patterns in Fig. **??**, write down a matrix $\mathbf{K}$ to transform some image represented by a 9-element vector, $\vec{x}$, into a 2-element vector, $\vec{y}$ ($\mathbf{K}\vec{x} = \vec{y}$).

(b) (2 points) The objects we will image first are 9-pixel representations of the Campanile and Cory Hall, shown in Fig. **??**. Write down two vectors, $\vec{x}_{campanile}$ and $\vec{x}_{cory}$ that represent the image data shown in Fig. **??**, using 0 for dark pixels and 1 for light pixels.

(c) (2 points) Use the $\mathbf{K}$ matrix from part a) to transform $\vec{x}_{campanile}$ and $\vec{x}_{cory}$ into 2-element vectors, $\vec{y}_{campanile}$ and $\vec{y}_{cory}$. (That is, compute $\mathbf{K}\vec{x}_{campanile} = \vec{y}_{campanile}$ and $\mathbf{K}\vec{x}_{cory} = \vec{y}_{cory}$.)

(d) (4 points) Based on your results in part c), how would you use the elements of your output vector $\vec{y}$ to distinguish between the Campanile and Cory Hall?

(e) (2 points) Now, let's consider using this method to look at other objects, like Sather Gate (arch-shaped) and the Greek Theater (bowl-shaped), which are represented in Fig. **??**.

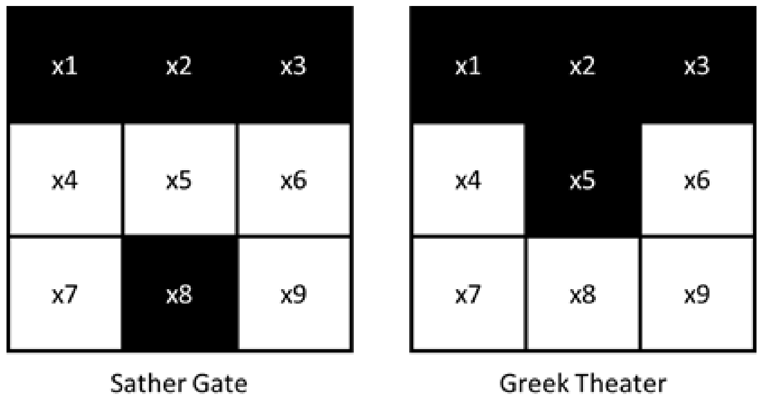


Figure 3: Sather Gate and the Greek Theater represented by 9 pixels

Write down $\vec{x}$ vectors representing these two objects, $\vec{x}_{sather}$ and $\vec{x}_{greek}$, and compute the output vectors $\vec{y}_{sather}$ and $\vec{y}_{greek}$ using the same mask patterns from Fig. **??**.

(f) (4 points) Can you use the outputs you computed in the previous part, $\vec{y}_{sather}$ and $\vec{y}_{greek}$, to distinguish between Sather Gate and the Greek Theater? Why or why not?

(g) (2 points) The difference between the Sather and Greek objects are the pixels x_5 and x_8. When we generate the vector, $\vec{y}$, the 5th and 8th elements of $\vec{x}$ select the 5th or 8th column of $\mathbf{K}$. In terms of linear independence, what is the relationship between the 5th and 8th column of $\mathbf{K}$?

(h) (2 points) You want to be able to distinguish all four objects with just two mask patterns, so you try one more thing: turning your illumination source 90° clockwise. This makes the new mask patterns shown in Fig. **??**.

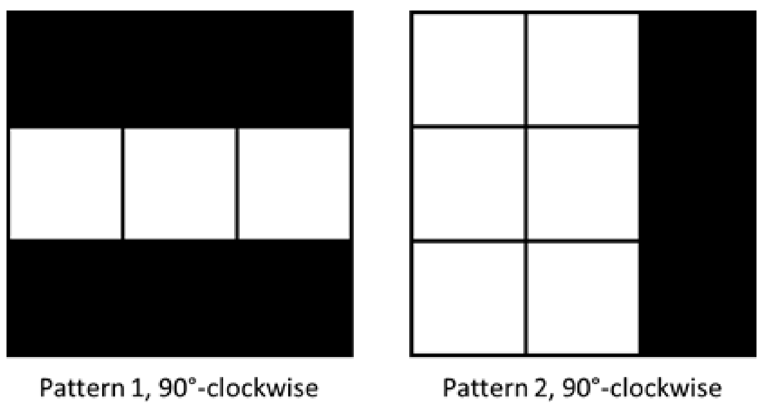


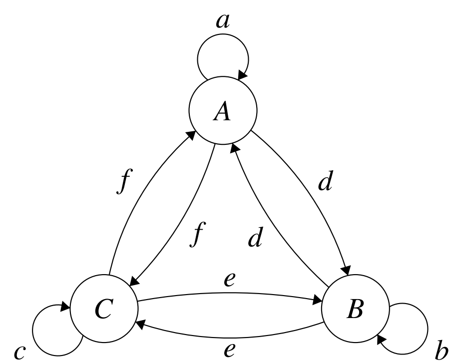
Figure 4: Two scanning patterns, rotated by 90°

Write down the new matrix, $\mathbf{K}_{90}$, representing the new mask patterns. In terms of linear independence, what is the relationship between the 5th and 8th column of $\mathbf{K}_{90}$? Use this knowledge to say whether or not you'll be able to distinguish between Sather Gate and the Greek Theater.

PRINT your name and student ID: ____________________

8. Reservoirs That Give and Take (15 points)

Consider a network of three water reservoirs A, B, and C. At the end of each day water transfers among the reservoirs according to the directed graph shown below.



The parameters a, b, and c—which label the self-loops—denote the *fractions* of the water in reservoirs A, B, and C, respectively, that stay in the same reservoir at the end of each day n. The parameters d, e, and f denote the fractions of the reservoir contents that transfer to adjacent reservoirs at the end of each day, according to the directed graph above.

Assume that the reservoir system is conservative—no water enters or leaves the system, which means that the total water in the network is constant. Accordingly, *for each node*, the weights on its self-loop and its two outgoing edges sum to 1; for example, for Node A, we have

$$a + d + f = 1,$$

and similar equations hold for the other nodes. Moreover, assume that all the edge weights are positive numbers—that is,

$$0 < a, b, c, d, e, f < 1.$$

The state evolution equation governing the water flow dynamics in the reservoir system is given by $s[n+1] = \mathbf{A}\,s[n]$, where the 3×3 matrix $\mathbf{A}$ is the state transition matrix, and $s[n] = \begin{bmatrix} s_A[n] & s_B[n] & s_C[n] \end{bmatrix}^T \in \mathbb{R}^3$ is the nonnegative state vector that shows the water distribution among the three reservoirs at the end of Day n, as fractions of the total water in the network.

In particular, $s[n] \succeq 0$ for all $n = 0, 1, 2, \ldots$, where the symbol $\succeq$ denotes componentwise inequality. Since the state vector represents the fractional distribution of water in the network, we have

$$\vec{1}^T s[n] = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} s_A[n] \\ s_B[n] \\ s_C[n] \end{bmatrix} = s_A[n] + s_B[n] + s_C[n] = 1 \quad \forall n = 0, 1, 2, \ldots.$$

(a) (5 points) Determine the state transition matrix $\mathbf{A}$.

(b) (5 points) Determine s^*, equilibrium state—that is, a state for which the following is true :

$$s[n+1] = s[n] = s^*$$

(c) (5 points) Suppose the state transition matrix for the network is given by

$$\mathbf{A} = \begin{bmatrix} 1/4 & 2/4 & 2/4 \\ 2/4 & 1/4 & 2/4 \\ 2/4 & 2/4 & 1/4 \end{bmatrix}.$$

Is it possible to determine the state $s[n]$ from the subsequent state $s[n+1]$? Provide a reasonably concise, yet clear and convincing explanation to justify your answer. You're NOT asked to compute $s[n]$ from $s[n+1]$ explicitly, but rather to assert, with justification, whether it is possible to do so.

9. Eigenvalues of Transition Matrices (15 points)

(a) (5 points) Show that a square matrix $\mathbf{A}$ and its transpose $\mathbf{A}^T$ have the same eigenvalues.

(b) (5 points) Let a square matrix $\mathbf{A}$ have rows which sum to one. Show that the vector $\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$ is an eigenvector of $\mathbf{A}$ and find its corresponding eigenvalue.

(c) (5 points) Show that a state transition matrix representing a *conservative* system will always have the eigenvalue $\lambda = 1$. Recall that all columns of a conservative state transition matrix sum to one.