University of California, Berkeley *Department of Physics* *Spring Semester 2004*

Physics 137B Sec. 2

MIDTERM EXAMINATION Thursday, March 4, 2004, 9:40-11:00 am

Advice

Please cross out any work which you do not wish to be graded. If your paper is neat, clear, and easy to read, it could affect your grade favorably.

Partial credit will be given for an incomplete or incorrect solution only for relevant, applicable statements that are logically presented. Random, disconnected comments will not be credited even if they happen to be correct. If you are unable to complete the answer to a question, please state clearly how far you got, and indicate how you would proceed to a solution.

Some potentially useful information is given below.

The maximum possible number of points is 60.

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1-D harmonic oscillator:
$x = (a + a^\dagger)/\sqrt{2\alpha}$, where $\alpha = (m\omega/\hbar)$ and the raising operator $a^\dagger$ and lowering operator a perform the following operations:

$$a^\dagger | n > = \sqrt{n+1} \; | n+1 >, \qquad a | n > = \sqrt{n} \; | n-1 >$$

Some integrals:

$$\int_0^\infty e^{-\eta x^2}\, dx = \frac{1}{2}\left(\frac{\pi}{\eta}\right)^{1/2} \qquad \int_0^\infty x e^{-\eta x^2}\, dx = \frac{1}{2\eta} \qquad \int_0^\infty x^2 e^{-\eta x^2}\, dx = \frac{1}{4\eta}\left(\frac{\pi}{\eta}\right)^{1/2}$$

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Midterm Solutions

1. (20 pts) Consider a harmonic oscillator with spring constant k:

$$H_0 = \frac{p^2}{2m} + \frac{1}{2}kx^2 .$$

Now suppose the spring constant increases slightly: $k \to k + \varepsilon$.

(a) Find the *exact* new energies, E_n, and expand the solution in a Taylor series in ε up to second order.

(b) Write down the perturbation Hamiltonian, H', in terms of a, a^+, and other quantities.

(c) Use Time Independent Perturbation theory to find the first order correction to the unperturbed energies E_n^0, and compare this to (a).

(d) Use Time Independent Perturbation theory to find the second order correction to the unperturbed energies E_n^0, and compare this to (a).

$$H = \frac{p^2}{2m} + \frac{1}{2}\underbrace{(k+\varepsilon)}_{k'}x^2 = \frac{p^2}{2m} + \frac{1}{2}k'x^2$$

$$\omega = \sqrt{\frac{k}{m}} \Rightarrow k = m\omega^2$$

$$\omega' = \sqrt{\frac{k'}{m}} = \sqrt{\frac{k+\varepsilon}{m}} = \sqrt{\frac{k}{m}}\sqrt{1+\varepsilon/k} = \omega\sqrt{1+\frac{\varepsilon}{k}}$$

a) $E_n^0 = \hbar\omega(n+\frac{1}{2}) = \hbar\sqrt{\frac{k}{m}}(n+\frac{1}{2})$

$$\boxed{E_n = \hbar\omega'(n+\tfrac{1}{2}) = \hbar\omega(n+\tfrac{1}{2})\sqrt{1+\varepsilon/k}}$$

$$f(x) = f(0) + \frac{1}{1!}f'(0)x + \frac{1}{2!}f''(0)x^2 + \dots$$

$f(x) = \sqrt{1+\varepsilon/k}$ $\quad f(0) = 1$

$f'(x) = \frac{1}{2k}(1+\varepsilon/k)^{-1/2}$ $\quad f'(0) = \frac{1}{2k}$ $\quad \Rightarrow \sqrt{1+\varepsilon/k} = 1 + \frac{\varepsilon}{2k} - \frac{1}{2}\frac{1}{4k^2} + \dots$

$f''(x) = \frac{-1}{4k^2}(1+\varepsilon/k)^{-3/2}$ $\quad f''(0) = \frac{-1}{4k^2}$

$$\boxed{E_n = \hbar\omega(n+\tfrac{1}{2})\left[1 + \tfrac{1}{2}\tfrac{\varepsilon}{k} - \tfrac{1}{8}\left(\tfrac{\varepsilon}{k}\right)^2 + \dots\right]}$$ where $\omega = \sqrt{\frac{k}{m}}$

b) $H' = H - H_0 = \left(\frac{p^2}{2m} + \frac{1}{2}(k+\varepsilon)x^2\right) - \left(\frac{p^2}{2m} + \frac{1}{2}kx^2\right) = \frac{1}{2}\varepsilon x^2$

$x = \frac{1}{\sqrt{2}\alpha}(a+a^+) = \sqrt{\frac{\hbar}{2m\omega}}(a+a^+)$ $\quad x^2 = \frac{\hbar}{2m\omega}(a+a^+)(a+a^+) = \frac{\hbar}{2m\omega}(a^2+aa^+ + a^+a + a^{+2})$

$$\boxed{H' = \frac{1}{2}\varepsilon x^2 = \frac{\hbar\varepsilon}{4m\omega}\left[a^2 + aa^+ + a^+a + a^{+2}\right]}$$

c) $E_n^{(1)} = \langle n|H'|n\rangle = \frac{\hbar\varepsilon}{4m\omega}\langle n|a^2 + aa^+ + a^+a + a^{+2}|n\rangle$

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$$= \frac{\hbar\varepsilon}{4m\omega}\left[\cancel{\langle n|a^2|n\rangle}^{0} + \langle n|aa^+|n\rangle + \langle n|a^+a|n\rangle + \cancel{\langle n|a^{+2}|n\rangle}^{0}\right]$$

$$= \frac{\hbar\varepsilon}{4m\omega}\left[\sqrt{n+1}\langle n|a|n+1\rangle + \sqrt{n}\langle n|a^+|n-1\rangle\right]$$

① c) cont...

$$E_n^{(1)} = \frac{\hbar\varepsilon}{4m\omega}\left[\sqrt{n+1}\sqrt{n+1} + \sqrt{n}+\sqrt{n}\right] = \frac{\hbar\varepsilon}{4m\omega}\left[2n+1\right] = \hbar\omega\left(n+\tfrac{1}{2}\right)\frac{\varepsilon}{2\underbrace{m\omega^2}_{k}}$$

$$\boxed{E_n^{(1)} = \hbar\omega\left(n+\tfrac{1}{2}\right)\left(\frac{1}{2}\frac{\varepsilon}{k}\right)}$$

The First order correction is exactly the $O(\varepsilon)$ term of the Taylor series of the exact energy, E_n, in (a).

d) $$E_n^{(2)} = \sum_{k\neq n}\frac{|\langle k|H'|n\rangle|^2}{E_n^{(0)}-E_k^{(0)}}$$

$$\langle k|H'|n\rangle = \frac{\hbar}{4m\omega}\varepsilon\langle k|a^2 + a^\dagger a + a a^\dagger + a^{\dagger 2}|n\rangle$$
$$= \frac{\hbar}{4m\omega}\varepsilon\left[\sqrt{n}\langle k|a|n-1\rangle + \sqrt{n}\langle k|a^\dagger|n-1\rangle + \sqrt{n+1}\langle k|a|n+1\rangle + \sqrt{n+1}\langle n|a^\dagger|n+1\rangle\right]$$
$$= \frac{\hbar}{4m\omega}\varepsilon\left[\sqrt{n}\sqrt{n-1}\underbrace{\langle k|n-2\rangle}_{\delta_{k,n-2}} + \sqrt{n}\sqrt{n}\underbrace{\langle k|n\rangle}_{\delta_{k,n}} + \sqrt{n+1}\sqrt{n+1}\underbrace{\langle k|n\rangle}_{\delta_{k,n}} + \sqrt{n+1}\sqrt{n+2}\underbrace{\langle n|n+2\rangle}_{\delta_{k,n+2}}\right]$$

$\Rightarrow$ There are only two nozero terms in $\sum_{k\neq n}$: $k=n-2$, $k=n+2$

$$E_n^{(2)} = \frac{|\langle n-2|H'|n\rangle|^2}{E_n^{(0)}-E_{n-2}^{(0)}} + \frac{|\langle n+2|H'|n\rangle|^2}{E_n^{(0)}-E_{n+2}^{(0)}} = \left(\frac{\hbar\varepsilon}{4m\omega}\right)^2\left[\frac{|\sqrt{n}\sqrt{n-1}|^2}{\hbar\omega(n+\frac{1}{2})-\hbar\omega((n-2)+\frac{1}{2})} + \frac{|\sqrt{n+1}\sqrt{n+2}|^2}{\hbar\omega(n+\frac{1}{2})-\hbar\omega(n+2+\frac{1}{2})}\right]$$
$$= \frac{\hbar^2\varepsilon^2}{16m^2\omega^2}\frac{1}{\hbar\omega}\left[\frac{n(n-1)}{2} + \frac{(n+1)(n+2)}{-2}\right] = \frac{\hbar\varepsilon^2}{32m^2\omega^3}\left[\cancel{n^2}-n-(\cancel{n^2}+3n+2)\right]$$

$$\boxed{E_n^{(2)} = -\frac{1}{8}\left(\frac{\varepsilon}{m\omega^2}\right)^2\hbar\omega\left[n+\tfrac{1}{2}\right]} = \hbar\omega\left(n+\tfrac{1}{2}\right)\left(-\frac{1}{8}\left(\frac{\varepsilon}{k}\right)^2\right)$$

The second order correction is exactly the $O(\varepsilon^2)$ term of the Taylor expansion of the exact energy, E_n in (a).

2. (20 pts) A particle of mass m is contained in the one-dimensional potential well

$$V(x) = \beta|x|.$$

Use the (normalized) gaussian wave function $\psi(x) = (2\alpha/\pi)^{1/4}\, e^{-\alpha x^2}$ as a trial function to show that the lowest upper bound on the ground state energy is

$$\frac{3}{2}\left(\frac{\hbar^2\beta^2}{2\pi m}\right)^{1/3}$$

Variational Therom: $E_0 \le \langle\psi|H|\psi\rangle$

$$H = \frac{p^2}{2m} + V(x) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \beta|x|$$

$$\frac{\partial\psi}{\partial x} = \frac{\partial}{\partial x}\left(\frac{2\alpha}{\pi}\right)^{1/4} e^{-\alpha x^2} = -2\alpha x\,\psi(x)$$

$$\frac{\partial^2\psi}{\partial x^2} = \left[-2\alpha + 4\alpha^2 x^2\right]\psi(x)$$

$$\langle\psi|H|\psi\rangle = \int_{-\infty}^{\infty}\psi^*_{(x)} H\psi(x)\,dx = \int_{-\infty}^{\infty}\psi^*(x)\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \beta|x|\right]\psi(x)\,dx$$

$$= \left(\frac{2\alpha}{\pi}\right)^{1/2}\int_{-\infty}^{\infty} e^{-\alpha x^2}\left[-\frac{\hbar^2}{2m}(-2\alpha + 4\alpha^2x^2)e^{-\alpha x^2} + \beta|x|\, e^{-\alpha x^2}\right]dx$$

$$= \left(\frac{2\alpha}{\pi}\right)^{1/2}\left[-\frac{\hbar^2}{2m}\left((-2\alpha)\int_{-\infty}^{\infty} e^{-2\alpha x^2}dx + 4\alpha^2\int_{-\infty}^{\infty} x^2 e^{-\alpha x^2}dx\right) + \beta\int_{-\infty}^{\infty}|x|\, e^{-2\alpha x^2}dx\right]$$

even functions $\Rightarrow \int_{-\infty}^{\infty} = 2\int_{0}^{\infty}$

$$= \left(\frac{2\alpha}{\pi}\right)^{1/2}\left[-\frac{\hbar^2}{2m}\left((-2\alpha)\, 2\underbrace{\int_0^{\infty} e^{-2\alpha x^2}}_{\frac{1}{2}\left(\frac{\pi}{2\alpha}\right)^{1/2}} + 4\alpha^2(2)\underbrace{\int_0^{\infty} x^2 e^{-2\alpha x^2}dx}_{\frac{1}{4(2\alpha)}\left(\frac{\pi}{2\alpha}\right)^{1/2}}\right) + 2\beta\underbrace{\int_0^{\infty} x\, e^{-2\alpha x^2}dx}_{\frac{1}{2(2\alpha)}}\right]$$

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$$= \left(\frac{2\alpha}{\pi}\right)^{1/2}\left[-\frac{\hbar^2}{2m}\left(-(2\alpha\pi)^{1/2} + \frac{1}{2}(2\pi\alpha)^{1/2}\right) + \frac{\beta}{2\alpha}\right]$$

$$\langle H\rangle = \frac{\hbar^2}{2m}\alpha + \frac{\beta}{(2\pi)^{1/2}}\frac{1}{\alpha^{1/2}}$$

Minimize $\langle H \rangle$: $\dfrac{\partial \langle H \rangle}{\partial \alpha} = \dfrac{\hbar^2}{2m} - \dfrac{1}{2}\dfrac{\beta}{(2\pi)^{1/2}}\dfrac{1}{\alpha^{3/2}} = 0$

$$\Rightarrow \quad \alpha = \left(\frac{\beta m}{\hbar^2 (2\pi)^{1/2}}\right)^{2/3}$$

$$\frac{\partial^2 \langle H \rangle}{\partial \alpha^2} = \frac{3}{4}\frac{\beta}{(2\pi)^{1/2}}\frac{1}{\alpha^{5/2}} > 0 \quad \text{for } \alpha > 0 \ \checkmark$$

$$\Rightarrow \langle H \rangle_{min} = \frac{\hbar^2}{2m}\left(\frac{\beta m}{\hbar^2 (2\pi)^{1/2}}\right)^{2/3} + \frac{\beta}{(2\pi)^{1/2}}\left(\left(\frac{\beta m}{\hbar^2 (2\pi)^{1/2}}\right)^{2/3}\right)^{-1/2}$$

$$= \frac{1}{2}\frac{\hbar^{2/3}\beta^{2/3}}{m^{1/3}(2\pi)^{1/3}} + \frac{\hbar^{2/3}\beta^{2/3}}{m^{1/3}(2\pi)^{1/3}}$$

$$= \frac{3}{2}\left(\frac{\hbar^2 \beta^2}{m(2\pi)}\right)^{1/3}$$

$$\Rightarrow \quad \boxed{E_0 \le \frac{3}{2}\left(\frac{\hbar^2 \beta^2}{2\pi m}\right)^{1/3}}$$

3. (20 pts) Consider a hydrogen atom in the n = 3 state.

Construct a five-column table labeled (left to right) ℓ, m_ℓ m_s, j and m_j.

(a) Assuming zero spin-orbit coupling:

(i) In the "ℓ-column" write down all the allowed values of ℓ.

(ii) In the "m_ℓ-column" write down the allowed values of m_ℓ corresponding to each value of ℓ.

(iii) In the "m_s-column" write down the allowed values of m_s corresponding to the ℓ and m_ℓ values.

(b) Now turn on the spin-orbit interaction so that the quantum numbers m_ℓ and m_s are replaced with j and m_j.

(i) In the last two columns, write down all the allowed values of j and m_j corresponding to the m_ℓ and m_s values.

(c) Finally, consider the hyperfine interaction in which the spin of the nucleus interacts with the total angular momentum of the electron. In a separate table, write down all the possible values of the quantum numbers f and m_f.

a) $n=3$ $\ell = 2,1,0$

$m_\ell = \ell, \ldots 0, \ldots -\ell$

$m_s = \pm 1/2$

b) $m_j = m_\ell + m_s$ $s = \frac{1}{2}$

$j = |\ell \pm s| \Rightarrow$ $\ell = 0$ $j = 1/2$

$\ell = 1$ $j = 1/2, 3/2$

$\ell = 2$ $j = 5/2, 3/2$

ℓ	m_ℓ	m_s	m_j	j
0	0	1/2	1/2	1/2
		-1/2	-1/2	
1	1	1/2	3/2	3/2
		-1/2	1/2	3/2, 1/2
	0	1/2	1/2	
		-1/2	-1/2	
	-1	1/2	-1/2	
		-1/2	-3/2	3/2
2	2	1/2	5/2	5/2
		-1/2	3/2	5/2, 3/2
	1	1/2	3/2	
		-1/2	1/2	
	0	1/2	1/2	
		-1/2	-1/2	
	-1	1/2	-1/2	
		-1/2	-3/2	
	-2	1/2	-3/2	
		-1/2	-5/2	5/2

c) $F = |j \pm i|$ $i = 1/2$

$j = 5/2, 3/2, 1/2$

$F = 3, 2, 1, 0$ $|M_F| \le F$

F	M_F
0	0
1	-1, 0, 1
2	-2, -1, 0, 1, 2
3	-3, -2, -1, 0, 1, 2, 3

(End)