Final Exam Solution: Problem 1

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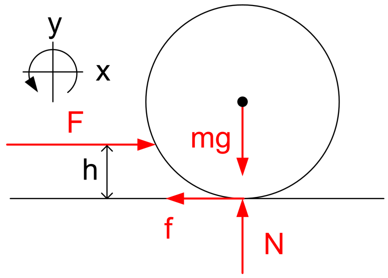
Part A



We use $F = ma$ and $\tau_{CM} = I_{CM}\alpha$ to find the linear and rotational acceleration. Since they are both constant, we can use the basic kinematics formulas to find the velocities.

$$\sum F = F - f = F - \mu_k mg = ma \quad \rightarrow \quad a = \frac{F}{m} - \mu_k g \tag{1}$$

$$v = v_0 + at = \left(\frac{F}{m} - \mu_k g\right) t \tag{2}$$

$$\sum \tau = F(r-h) - fr = I\alpha = \frac{2}{5}mr^2\alpha \quad \rightarrow \quad \alpha = \frac{5}{2mr^2}\left[F(r-h) - \mu_k mgr\right] \tag{3}$$

$$\omega = \omega_0 + \alpha t = \frac{5t}{2mr^2}\left[F(r-h) - \mu_k mgr\right] \tag{4}$$

We know that at time t_0, the angular and linear speeds are related by $\omega_0 = 2v_0/r$. Thus, we plug (2) and (4) into this relationship.

$$\omega_0 = \frac{2v_0}{r} \quad \rightarrow \quad \frac{5t}{2mr^2}\left[F(r-h) - \mu_k mgr\right] = \frac{2}{r}\left(\frac{F}{m} - \mu_k g\right) t \quad \rightarrow$$
$$F - F\frac{h}{r} - \mu_m g = \frac{4}{5} - \frac{4}{5}\mu_k mg \quad \rightarrow \quad \frac{1}{5}\left(F - \mu_k mg\right) = F\frac{h}{r} \quad \rightarrow \quad \boxed{h = \frac{r}{5} - \frac{\mu_k mgr}{5F}} \tag{5}$$

Grading Rubric. This part was worth 10 points, and broken down as follows: +2 for the FBD; +4 for the force and torque equations; +2 for finding $v(t)$ and $\omega(t)$; +2 for solving for h.

Part B: Force Method

Kinematics. There is now no cue force, but the friction is the same as in the FBD above. We use force and torque (about the CM) to find (constant) accelerations, which we use to calculate velocities.

$$\sum F = -f = -\mu_k mg = ma \quad \rightarrow \quad a = -\mu_k g \quad \rightarrow \quad v = v_0 - \mu_k gt \tag{6}$$

$$\sum \tau = -fr = \frac{2}{5}mr^2\alpha \quad \rightarrow \quad \alpha = -\frac{5\mu_k g}{2r} \quad \rightarrow \quad \omega = \omega_0 - \frac{5}{2r}\mu_k gt \tag{7}$$

Determining the Final Direction of v. Let us assume that v reaches zero before ω reaches $-v/r$. (In other words, this would be the case when the ball switches directions an rolls backwards.)

$$v: \quad 0 = v_0 - \mu_k g t_0 \quad \rightarrow \quad t_0 = \frac{v_0}{\mu_k g}; \qquad \omega: \quad 0 = \frac{2v_0}{r} - \frac{5}{2r}\mu_k g t_0 \quad \rightarrow \quad t_0 = \frac{4}{5}\frac{v_0}{\mu_k g} \tag{8}$$

Clearly, the angular velocity reaches zero before the linear one does, which indicates that the final direction of $\mathbf{v}$ is to the right. At some time t^*, we have the rolling without slipping condition $v = -\omega r$. (The negative sign is there because counter-clockwise rotation was defined to be positive.) Applying this condition at t^*,

$$v = v_0 - \mu_k g t^* = -\omega r = -\left(2v_0 - \frac{5}{2}\mu_k g t^*\right) \rightarrow \frac{7}{2}\mu_k g t^* = 3v_0 \rightarrow t^* = \frac{6v_0}{7\mu_k g} \tag{9}$$

$$v_{final} = v\left(t^*\right) = v_0 - \mu_k g t^* = v_0 - \mu_k g\left(\frac{6v_0}{7\mu_k g}\right) = v_0 - \frac{6}{7}v_0 = \boxed{\frac{1}{7}v_0} \tag{10}$$

Grading Rubric. This part was worth 10 points and broken down as follows: +2 for force and torque equations; +1 for velocities as functions of time; +3 for proving that the final velocity is to the right; +2 for imposing rolling without slipping; +2 for solving the final velocity's magnitude.

Part B: Angular Momentum Method

Choose a point O on the table. There is only one force (friction) acting on the ball after the initial hit. The friction acts on point P, which is also in contact with the table. Therefore, $\mathbf{r} = \overrightarrow{OP}$ is parallel to the table. The frictional force $\mathbf{f}$ is also parallel to the table. The torque due to friction is $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{f} = 0$ because the cross product of two parallel vectors is zero. Because $\boldsymbol{\tau} = \frac{d\mathbf{L}_O}{dt}$ and $\boldsymbol{\tau} = 0$, we know that $\mathbf{L}_O$ is a constant, i.e., the angular momentum about O is conserved. This momentum has two components: one due to the spinning of the ball, and one due to the fact the CM "rotates" about O.

$$\mathbf{L}_{O,init} = \bar{\mathbf{L}} + \bar{\mathbf{r}} \times m\bar{\mathbf{v}} = \left(\frac{2}{5}mr^2\frac{2v_0}{r} - mrv_0\right)\mathbf{k} = -\frac{1}{5}mrv_0\mathbf{k} \tag{11}$$

We note that the angular momentum is negative, meaning that when rolling without slipping sets in, the rotation will be negative, which is defined as clockwise. This implies the final velocity will be to the right.

We now express the final angular momentum (dropping the vector $\mathbf{k}$ for convenience), keeping in mind the rolling without slipping condition $v_f = -\omega_f r$.

$$L_{O,final} = \frac{2}{5}mr^2\left(\frac{-v_f}{r}\right) - mrv_f = -\frac{7}{5}mrv_f \tag{12}$$

Applying the conservation of angular momentum,

$$L_{O,init} = L_{O,final} \quad \rightarrow \quad -\frac{1}{5}mrv_0 = -\frac{7}{5}mrv_f \quad \rightarrow \quad v_f = \boxed{\frac{1}{7}v_0} \tag{13}$$

Grading Rubric. This part is worth 10 points, broken down as follows: +3 calculation of initial angular momentum; +3 calculation of final angular momentum; +3 direction of final velocity; +1 using ang. mo. conservation to calculate final velocity.

2

a.) $\rho g h_c + \frac{1}{2}\cancel{\rho v_c^2} + \cancel{P_c} = \rho g h_2 + \frac{1}{2}\rho v_2^2 + \cancel{P_2}$; $P_c = P_2 = 1\,atm$ and $v_c = 0$, $h_c - h_2 = h$

$$\cancel{\rho} g h = \frac{1}{2}\cancel{\rho} v_2^2$$

$$\boxed{\sqrt{2gh} = v_2}$$

$$A_1 v_1 = A_2 v_2$$

$$v_1 = \frac{A_2}{A_1} v_2 = \frac{\sqrt{2gh}}{4} = \boxed{\sqrt{\frac{gh}{8}} = v_1}$$

b.) $\boxed{P_2 = 1\,atm}$ b/c exposed to air. Can also use Bernoulli's w/ soln to 2a

$$P_1 + \cancel{\rho g h_1} + \frac{1}{2}\rho v_1^2 = P_2 + \cancel{\rho g h_2} + \frac{1}{2}\rho v_2^2 \qquad h_2 = h_1$$

$P_2 = P_{atm} \approx 101 \times 10^3\,Pa = 1\,atm$

$$P_1 = P_{atm} + \frac{1}{2}\rho(v_2^2 - v_1^2)$$

$$P_1 = P_{atm} + \frac{1}{\cancel{2}}\rho\left(\cancel{2}gh - \frac{\cancel{2}gh}{16}\right)$$

$$\boxed{P_1 = P_{atm} + \frac{15}{16}\rho g h}$$

c.) $A_c v_c = A_2 v_2$

$$v_c = \frac{A_2 v_2}{A_c}$$

$$v_c = \frac{\sqrt{2gh}}{100}$$

$$v_c = -\frac{dh}{dt} \Rightarrow -\frac{\partial h}{\partial t} = \frac{\sqrt{2gh}}{100}$$

$$\int_{h_0}^{h} \frac{dh}{\sqrt{h}} = -\int_0^t \frac{\sqrt{2g}}{100}\,dt$$

$$2(\sqrt{h} - \sqrt{h_0}) = \frac{-\sqrt{2g}}{100}t$$

$$\sqrt{h} = \frac{-\sqrt{2g}}{200}t + \sqrt{h_0}$$

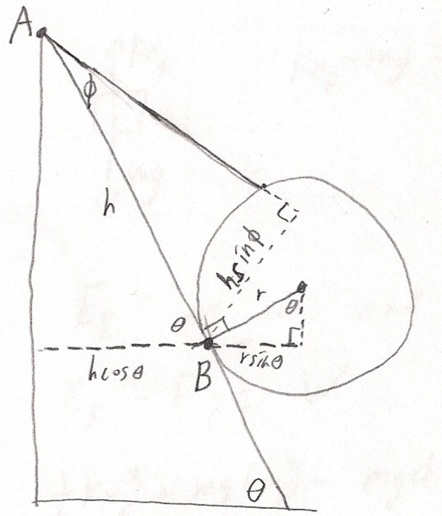
now solve for $h = 0$; $h_0 = \frac{1}{10}\,m$

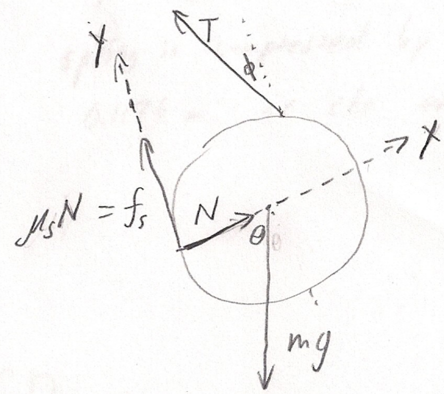
so

$$0 = \frac{-\sqrt{2g}}{200}t + \sqrt{\frac{1}{10}}$$

$$t = \frac{200}{\sqrt{2g}} \cdot \frac{1}{\sqrt{10}} \approx 14.29\,\text{secs.}$$

#3





$$\Sigma F_Y = \mu_s N + T\cos\phi - mg\sin\theta = 0 \qquad (1)$$

$$\Sigma F_X = N - T\sin\phi - mg\cos\theta = 0 \qquad (2)$$

$$\Sigma \tau_B = T h\sin\phi - mgr\sin\theta = 0 \qquad (3)$$

(Alternatively use $\Sigma\tau_A = Nh - mg(h\cos\theta + r\sin\theta) = 0$)

$$(3):\ \boxed{T = \frac{mgr\sin\theta}{h\sin\phi}}$$

$$(1):\ \mu_s N = mg\sin\theta - T\cos\phi = mg\sin\theta - \frac{mgr}{h}\frac{\sin\theta\cos\phi}{\sin\phi}$$

$$(2):\ N = T\sin\phi + mg\cos\theta = \frac{mgr\sin\theta\cancel{\sin\phi}}{h\cancel{\sin\phi}} + mg\cos\theta$$

$$\mu_s = \frac{(1)}{(2)} = \frac{\cancel{mg\sin\theta}\left(1 - \frac{r\cos\phi}{h\sin\phi}\right)}{\cancel{mg\sin\theta}\left(\frac{r}{h} + \frac{\cos\theta}{\sin\theta}\right)} = \boxed{\frac{(h - r\cot\phi)}{(r + h\cot\theta)}}$$

Problem 4

(a) $E_k = \frac{1}{2}k(\Delta x_0)^2 = \frac{1}{2}\times 3\times 0.3^2 = 0.135\,(J)$

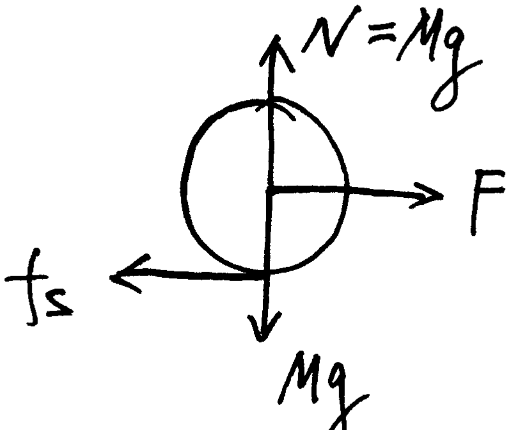
$E_{k,t} = \frac{1}{2}Mv^2$

$E_{k,r} = \frac{1}{2}I\omega^2 = \frac{1}{2}\left(\frac{1}{2}Mr^2\right)\omega^2 = \frac{1}{4}Mv^2$

$$\begin{cases} E_{k,t}/E_{k,r} = 2 \\ E_{k,t} + E_{k,r} = E_k = 0.135\,(J) \end{cases}$$

$$\therefore \begin{cases} E_{k,t} = 0.09\,(J) \\ E_{k,r} = 0.045\,(J) \end{cases}$$

(b)



$$\begin{cases} F - f_s = Ma \\ rf_s = I\alpha = \frac{1}{2}Mr^2\alpha \\ r\alpha = a \end{cases}$$

$$\Downarrow$$

$$f_s = \frac{1}{3}F$$

$$F_{net} = F - f_s = \frac{2}{3}F = \frac{2}{3}(-kx)$$

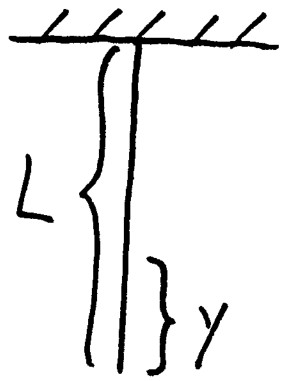
$$\therefore \quad a = \frac{F_{net}}{M} = -\frac{2k}{3M}x \implies \ddot{x} = -\frac{2k}{3M}x$$

$$\omega = \sqrt{\frac{2k}{3M}}, \quad T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{3M}{2k}}.$$

If you plug in the numerical value for k
you should get $T = 2\pi\sqrt{\frac{M}{2}} = 4.44\sqrt{M}$

Problem 5

(a)



At position y, we have

$$T(y) = \mu y g$$

$T(y)$ is the tension in the rope at position y, and μ is the mass in per unit length.

Thus the wave velocity (for transverse waves) is

$$V = \sqrt{\frac{T(y)}{\mu}} = \sqrt{gy}$$

(b) $\quad V(y) = \sqrt{gy} \implies \frac{dy}{dt} = \sqrt{gy}$

$$\implies dt = \frac{dy}{\sqrt{gy}} \implies \int_0^{t_0} dt = \int_0^L \frac{dy}{\sqrt{gy}}$$

$$\therefore\ t_0 = 2\sqrt{\frac{L}{g}}$$

Problem 6

$$\begin{cases} h_1 = 0.125\,m \\ h_2 = 0.395\,m \end{cases}$$

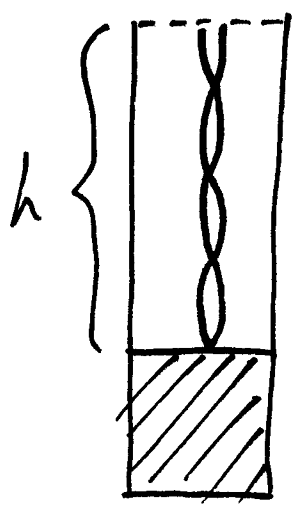
we have no resonance between h_1 and h_2, so we have

$$\frac{\lambda}{2} = h_2 - h_1$$

$$\therefore \lambda = 2(h_2 - h_1) = 2 \times 0.27 = 0.54\,(m)$$

$$f = \frac{V}{\lambda} = \frac{343}{0.54} = 635\,(Hz)$$

The picture is roughly (although not strictly) like the way below

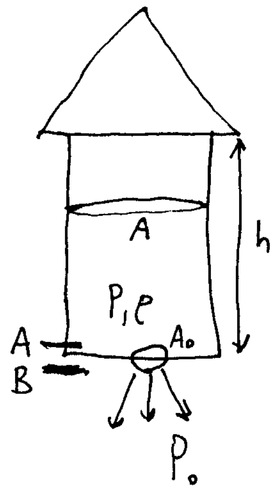


Problem 7 Solution

a) Bernoulli's Equation is: $P_i + \frac{1}{2}\rho_i v_i^2 + \rho_i g y_i = \text{constant}$

Apply Bernoulli's Equation at both sides of the output hole

$$\Rightarrow P_A + \frac{1}{2}\rho_A v_A^2 + \rho_A g y_A = P_B + \frac{1}{2}\rho_B v_B^2 + \rho_B g y_B$$



Now, $P_A = P$

$P_B = P_0$

$\rho_A \approx \rho_B = \rho$

$v_A \approx 0$

$y_A \approx y_B$ (taken immediately in front of and behind output hole)

Then

$$\Rightarrow P + \frac{1}{2}\rho \cancelto{0}{v_A^2} + \rho g y_A = P_0 + \frac{1}{2}\rho v_B^2 + \rho g y_B$$

$$\Rightarrow \frac{1}{2}\rho v_B^2 = P - P_0 + \rho g \cancelto{0}{\underbrace{(y_A - y_B)}_{\approx 0}}$$

$$\boxed{\Rightarrow v_B = \sqrt{\frac{2(P-P_0)}{\rho}}}$$ — emission speed of the propelling gases

b) From the momentum-impulse theorem, $F = \frac{dp}{dt}$.

The force on the exiting gas by the "rocket" is $F_g = \frac{dp}{dt} = v_B \frac{dm}{dt}$.

$\frac{dm}{dt}$ is the mass flow out of the rocket: $\frac{dm}{dt} = \rho \frac{dV}{dt} = \rho \underbrace{A_0 v_B}_{\text{volume flow out of the rocket}}$

Therefore, $F_g = \frac{dp}{dt} = v_B \frac{dm}{dt} = v_B(\rho A_0 v_B) = \rho A_0 v_B^2$

From Newton's Third Law, the force the gas applies on the "rocket" is equal and opposite to the force the "rocket" applies on the gas. Therefore... $\vec{F}_g = -\vec{F}_r$, or $|\vec{F}_g| = |\vec{F}_r|$

Then, $|\vec{F}_r| = |\vec{F}_g| = \rho A_0 v_B^2$

Substituting v_B from part a), $\Rightarrow |\vec{F}_r| = \rho A_0 \left(\sqrt{\frac{2(P-P_0)}{\rho}}\right)^2 = \cancel{\rho} A_0 \cdot \frac{2(P-P_0)}{\cancel{\rho}}$

$$\Rightarrow |\vec{F}_r| = \boxed{2A_0(P-P_0)}$$ — thrust force on the rocket due to the emitted gases