CS 70 Discrete Mathematics and Probability Theory
Fall 2017 Rao — Midterm 1

PRINT Your Name: ________________________________, ________________________________
(last) (first)

SIGN Your Name: ____

PRINT Your Student ID: ____

CIRCLE your exam room:
Hearst Field Annex A1 Kroeber 160 Moffitt 101 Morgan 101 Mulford 159 Pimentel 1 Other

Name of the person sitting to your left: ____

Name of the person sitting to your right: ____

- After the exam starts, please *write your student ID (or name) on every odd page* (we will remove the staple when scanning your exam).
- We will not grade anything outside of the space provided for a problem unless we are clearly told in the space provided for the question to look elsewhere.
- The questions vary in difficulty, so if you get stuck on any question, it might help to leave it and try another one.
- On questions 1-2: You need only give the answer in the format requested (e.g., true/false, an expression, a statement.) Note that an expression may simply be a number or an expression with a relevant variable in it. **For short answer questions, correct clearly identified answers will receive full credit with no justification. Incorrect answers may receive partial credit.**
- On question 3-4, do give arguments, proofs or clear descriptions as requested.
- You may consult only *one sheet of notes*. Apart from that, you may not look at books, notes, etc. Calculators, phones, computers, and other electronic devices are NOT permitted.
- There are **9** single sided pages on the exam. Notify a proctor immediately if a page is missing.
- **You may, without proof, use theorems and lemmas that were proven in the notes and/or in lecture.**
- **You have 120 minutes: there are 4 questions (with 40 parts) on this exam worth a total of 115 points.**

Do not turn this page until your instructor tells you to do so.

SID: ____

1. **TRUE or FALSE?: 45 points, each part 3 points**
For each of the questions below, answer TRUE or FALSE.
Clearly indicate your correctly formatted answer: this is what is to be graded. No need to justify!

   1. $(P \implies \neg Q) \equiv (Q \implies \neg P)$
   2. $\forall x \in S$, $[P(x) \vee Q(x)]$ is equivalent to $[\forall x \in S, P(x)] \vee [\forall x \in S, Q(x)]$.
   3. $(\neg \forall n \in N, P(n) \implies P(n+1)) \equiv \exists n \in N, \neg P(n)$.
   4. In the stable marriage problem a female can only get her optimal partner in the female optimal pairing.
   5. Consider a botched execution of the stable marriage algorithm: one shy man skips the first person on his list, but the algorithm still terminates. There is no rogue couple in this pairing.
   6. Consider an algorithm for stable room-mates (or single gender) problem where we start with a pairing and continuing to pair any rogue couple. If this algorithm terminates we get a stable pairing.
   7. The following statement is a proposition:
   "A planar graph requires at least six colors to be colored."
   8. There exists an undirected graph on n vertices such that no two vertices have the same degree.
   9. Any graph with a vertex of degree d can be $d+1$ colored.
   10. Maximum degree of any planar graph is 6.
   11. If a triangulated planar graph can be 4 colored then all planar graphs can be 4 colored.
   12. The number of edges to cut a 4 dimensional hypercube in half is less than the number of edges to cut K_6 in half. (To cut in half means the removing the edges to leave to connected components of exactly the same size. We are asking about the size of the removed set of edges.)
   13. Any degree 4 graph is planar since it obeys Euler's formula.
   14. If $\gcd(x,y) = d$ and $gcd(y,z) = c$, then $gcd(x,z) \geq gcd(c,d)$.
   15. If $\gcd(x,y) = d$ and $gcd(y,z) = c$, then $gcd(x,z) \leq gcd(c,d)$.

2. An expression or number: 3/3/3/3/3/3/8/8 points. Clearly indicate your correctly formatted answer: this is what is to be graded. No need to justify!

1. Recall that a bipartite graph is a graph, $G = (V,E)$ where $V = L \cup R$ and $E \subseteq L \times R$, or where edges connect vertices in L with R. What is the sum of degrees of vertices in L in terms of $|L|$,$|R|$, and $|E|$? (**Answer is an expression or number.**)

2. A connected simple graph, $G = (V,E)$ has one more vertex than it has edges. How many faces are there in a planar drawing of this graph.
(**Answer is an expression or number.**)

3. What is the maximum number of connected components for a graph with $n \geq 3$ vertices if it has more than $(n-1)(n-2)/2$ edges.(**Answer: an expression or number.**)

4. Consider a planar graph G where each face is incident to exactly 5 edges. Derive an expression for the number of edges in G, e, in terms of the number of vertices, v in G. (**Answer is an expression or number.**)

5. How **many solutions** to $5x = 10 \pmod{30}$? (**Answer is a number.**)

6. What are the last three digits of the product $9 \times 99 \times 999 \times \ldots \times 999999999$? (**Answer is a number.**)

7. Given any graph G that has a k-coloring. Give a short argument for the correctness of your upper bound. (We do note that a really bad upper bound, say $|V|$ is not an interesting answer in terms of credit. The expressions may involve the variable k, the number of vertices and edges in G as well as other introduced variable.)

   (a) Provide an upper bound on the maximum number of colors that could be required if one removes an edge from G. (**Answer is an expression or number.**)

   (b) Provide an upper bound on the maximum number of colors one needs to color the graph if one adds an edge between two vertices in G. (**Answer is an expression or number.**)

   (c) Provide an upper bound on the maximum number of colors one needs to color the graph if one adds ℓ edges between vertices in G where the edges form a cycle. That is, the edges when viewed without any edges in G form a cycle. (**Answer is an expression or number.**)

   (d) Provide an upper bound on the maximum number of colors one needs to color the graph if one adds ℓ edges between vertices in G where the edges form a tree. That is, the edges when viewed without any edges in G form a tree on the incident vertices. (**Answer is an expression.**)

8. Consider the graph G formed with vertex set $V = \{0,\dots,m-1\}$, and edge set $E = \{(x,y) : y = x+g \pmod{m}\}$. Let $d = gcd(g,m)$, and $g \neq 0$.

   (a) What is the maximum degree of any vertex in the graph? (**Answer is an expression.**)

   (b) What is the length of the shortest cycle? (**Answer is an expression.**)

9. (a) Give an example of a stable marriage instance with 2 men and 2 women where there are two different stable pairings. (**Answer is a description of an instance.**)

   (b) Give an example of a stable marriage instance with at least three different stable pairings. (**Answer: You may use the table, but if you can describe a construction, that is fine too. Either list three stable pairings or argue they exist.**)

	Women			
A	1	2	3	4
B	2	1	4	3
C	3	4	2	1
D	4	3	1	2
	Men			
1				
2				
3				
4				

3. Some Proofs: 5/5/6/6 points

1. Prove the statement: If $n^2 - 1$ is not divisible by 3, then $n - 1$ is not divisible by 3.

2. Prove that if one directs the edges in a tree there is at least one vertex with zero in-degree, and one vertex that has zero out-degree.

3. Prove that the edges of any planar graph can be directed so that every vertex has at most out-degree 3. (Strengthen Inductive Claim: Prove that one can direct the edges of every planar graph drawing so that every vertex has out degree at most 3 and the vertices on the exterior face have out degree 2.)

4. We wish to prove that $\frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n-1}{2n} \leq \frac{1}{\sqrt{3n}}$ for $n \geq 1$.
   Base case: $\frac{1}{2} \leq \frac{1}{\sqrt{3}}$ is true since $\sqrt{3} \leq 2$.
   Induction Hypothesis: $\frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n-1}{2n} \leq \frac{1}{\sqrt{3n}}$.
   Induction Step:
   By the induction hypothesis, we have

$$\frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n+1}{2n+2} \leq \frac{1}{\sqrt{3n}} \cdot \frac{2n+1}{2n+2}$$

   To finish, we could show that

$$\frac{1}{\sqrt{3n}} \cdot \frac{2n+1}{2n+2} \leq \frac{1}{\sqrt{3n+3}}.$$

   Rearranging, we need

$$\left(\frac{2n+1}{2n+2}\right)^2 \leq \frac{3n}{3n+3},$$

   using $\frac{2n+1}{2n+2} = 1 - \frac{1}{2n+2}$ and $\frac{3n}{3n+3} = 1 - \frac{3}{3n+3}$, we obtain

$$(1 - \frac{1}{2n+2})^2 \leq 1 - \frac{1}{n+1} + \frac{1}{4(n+1)^2} \leq 1 - \frac{3}{3(n+1)}. \tag{1}$$

   Removing the 1 and multiplying through by $12(n+1)^3$, we obtain

$$-12(n+1)^2 + 3(n+1) \leq -12(n+1)^2 \tag{2}$$

   But, this suggests that we need $3(n+1) \leq 0$. This clearly does not work.
   Strengthen the claim and carry out the induction proof. Your answer should clearly state the new claim, and clearly identify new versions of inequalities 1 and 2. (Hint: $\frac{1}{3n+1} \leq \frac{1}{3n}$ for $n > 0$)

4. Stable trios: 8 points.

Suppose there are n men, n women, and n pet dogs that we want to group into trios of one man, woman, and dog each. The men and women have preference lists for each other as in the usual stable marriage problem. In addition, the men and dogs have preference lists for each other. However, each woman has a set of dogs that she hates.

We want to find a way to group the men, women, and dogs into trios of one man, one woman, and one dog each such that the following stability criteria hold:

1. No man and dog not in the same trio prefer each other to their dog and man in their respective trios
2. Each man is in a trio with the best dog he can get in any trio satisfying condition 1
3. No man and woman not in the same trio prefer each other to their woman and man in their respective trios
4. No woman is in a trio with a dog that she hates

Show how you can find a stable trio if one exists, and otherwise determine that there is no stable trio.