Math 54. Solutions to the Final Exam

1. (20 points) Let (x_1, x_2, x_3) be the solution to the linear system

$$\begin{aligned} x_1 + 2x_2 + 7x_3 &= 6 \\ 2x_1 + x_2 &= 4 \\ -x_1 + 3x_2 + 5x_3 &= 0\,. \end{aligned}$$

Use Cramer's rule to find x_3.

By Cramer's rule,

$$x_3 = \frac{\det A_3(\vec{b})}{\det A} = \frac{\begin{vmatrix} 1 & 2 & 6 \\ 2 & 1 & 4 \\ -1 & 3 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 7 \\ 2 & 1 & 0 \\ -1 & 3 & 5 \end{vmatrix}} = \frac{(-1)\begin{vmatrix} 2 & 6 \\ 1 & 4 \end{vmatrix} - 3\begin{vmatrix} 1 & 6 \\ 2 & 4 \end{vmatrix}}{-2\begin{vmatrix} 2 & 7 \\ 3 & 5 \end{vmatrix} + \begin{vmatrix} 1 & 7 \\ -1 & 5 \end{vmatrix}}$$

$$= \frac{-(8-6) - 3(4-12)}{-2(10-21) + (5+7)} = \frac{-2+24}{22+12} = \frac{22}{34} = \frac{11}{17}\,.$$

2. (30 points) Let $A = \begin{bmatrix} 2 & 4 & 3 & 1 & 17 \\ 1 & 2 & 0 & 1 & 2 \\ 2 & 4 & 1 & 2 & 8 \\ 2 & 4 & 2 & -5 & 19 \end{bmatrix}$. and $B = \begin{bmatrix} 1 & 2 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Given that A is row equivalent to B, and using the methods taught in Math 54, find:

(a). A basis for $\operatorname{Row} A$.

Use the nonzero rows of B:

$$\begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 4 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}.$$

(b). A basis for $\operatorname{Col} A$.
Use the pivot columns of A:

$$\begin{bmatrix} 2 \\ 1 \\ 2 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ 2 \\ -5 \end{bmatrix}.$$

(c). A basis for $\operatorname{Nul} A$.

Further reduce B to reduced row echelon form, and write the solution of $A\vec{x} = \vec{0}$ in parametric vector form:

$$\begin{bmatrix} 1 & 2 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2x_2 - 3x_5 \\ x_2 \\ -4x_5 \\ x_5 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ -4 \\ 1 \\ 1 \end{bmatrix} .$$

Therefore a basis for $\operatorname{Nul} A$ is

$$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -3 \\ 0 \\ -4 \\ 1 \\ 1 \end{bmatrix} .$$

3. (20 points) Given bases

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\} \qquad \text{and} \qquad \mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} ,$$

find a matrix M such that

$$[\vec{x}]_{\mathcal{B}} = M\,[\vec{x}]_{\mathcal{C}}$$

for all $\vec{x} \in \mathbb{R}^2$.

By the method of Example 3 on page 230, we compute $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$:

$$\begin{aligned} \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{c}_1 & \vec{c}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 1 \end{bmatrix} &\sim \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{bmatrix} \\ \sim \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{bmatrix} &\sim \begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & 2 & 1 \end{bmatrix} , \end{aligned}$$

and so

$$M = \underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix} .$$

4. (20 points) Let V be $\mathbb{P}_2$, with the inner product

$$\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx\ .$$

Compute the orthogonal projection of p onto the subspace spanned by q, where

$$p(x) = x^2 \qquad \text{and} \qquad q(x) = 1 + x\ .$$

We have

$$\langle p, q\rangle = \int_0^1 x^2(1+x)\,dx = \int_0^1 (x^3 + x^2)\,dx = \frac{x^4}{4} + \frac{x^3}{3}\bigg|_0^1 = \frac{1}{4} + \frac{1}{3} = \frac{7}{12}$$

and

$$\langle q, q\rangle = \int_0^1 (1+x)^2\,dx = \int_0^1 (x^2 + 2x + 1)\,dx = \frac{x^3}{3} + x^2 + x\bigg|_0^1 = \frac{1}{3} + 1 + 1 = \frac{7}{3}\ .$$

Therefore

$$\operatorname{proj}_q p = \frac{\langle p, q\rangle}{\langle q, q\rangle} q = \frac{7/12}{7/3}(1+x) = \frac{1+x}{4}\ .$$

5. (20 points) Find all least-squares solutions to the linear system

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 0 \\ x_1 \qquad\quad - x_3 &= 1 \\ x_2 + x_3 &= 1\ . \end{aligned}$$

We have

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \qquad \text{and} \qquad \vec{b} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\ ,$$

so

$$A^T A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 3 \\ 0 & 3 & 3 \end{bmatrix} \qquad \text{and} \qquad A^T\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\ .$$

Therefore the normal equations have the following augmented matrix, which is row reduced as follows:

$$\begin{bmatrix} 2 & 2 & 0 & 1 \\ 2 & 5 & 3 & 1 \\ 0 & 3 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & 2 & 0 & 1 \\ 0 & 3 & 3 & 0 \\ 0 & 3 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & 2 & 0 & 1 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & \frac{1}{2} \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & \frac{1}{2} \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\ .$$

Therefore the solutions are

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 + \frac{1}{2} \\ -x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

(in parametric vector form).

6. (20 points) For each of the parts listed below, either *give an example of such a matrix,* or give a brief reason why *no example exists.* If you give an example, it must be either a specific matrix, or a matrix expression (involving sums, products, inverses, etc.) that evaluates to a specific matrix.

(a). A 3×3 matrix A with eigenvalues 1, 2, and 3, and corresponding eigenvectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}$, respectively.

The matrix $[\,\vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3\,]$ is nonsingular (it is upper triangular, so it is easy to see that its determinant is nonzero). Therefore, a matrix A exists:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{bmatrix}^{-1}$$

(b). Same as part (a), but with $\vec{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

Since $\vec{v}_1 + \vec{v}_2 - \vec{v}_3 = \vec{0}$, the three vectors are linearly dependent. Since eigenvectors for distinct eigenvalues must be linearly independent (Theorem 2 on page 240), there can be no such matrix A.

(c). Same as part (a), but with $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$, and requiring that A be *symmetric.*

For symmetric matrices, eigenvectors of distinct eigenvalues must be orthogonal (Theorem 1 on page 341). However, $\vec{v}_1$ and $\vec{v}_3$ are not orthogonal, so there can be no such matrix A.

7. (25 points) (a). Compute the Wronskian $W[x, e^x, \sin x]$.

$$W[x, e^x, \sin x] = \begin{vmatrix} x & e^x & \sin x \\ 1 & e^x & \cos x \\ 0 & e^x & -\sin x \end{vmatrix} = x \begin{vmatrix} e^x & \cos x \\ e^x & -\sin x \end{vmatrix} - \begin{vmatrix} e^x & \sin x \\ e^x & -\sin x \end{vmatrix}$$

$$= xe^x(-\sin x - \cos x) + 2e^x \sin x = e^x((2 - x)\sin x - x\cos x)\,.$$

Note: Wrong answer.
The Wronskien Lemma states that the functions are linearly independent because you can find at least one case where the Wronskien is nonzero.

(b). Are the functions x, e^x, $\sin x$ linearly independent? Explain, using the Wronskian.

No, because the Wronskian is not everywhere zero; for example,

$$W[x, e^x, \sin x](\pi) = \pi e^\pi \ .$$

(c). Use a property of Wronskians to show that there is no differential equation

$$y''' + p_1 y'' + p_2 y' + p_3 y = 0 \ ,$$

with p_1, p_2, and p_3 continuous on $(-\infty, \infty)$, for which x, e^x, and $\sin x$ are all solutions.

This is a Wronskian of three functions, so if those functions are solutions to the given (third-order) differential equation, then the Wronskian would either always be zero or always be nonzero (Theorem 3 on page 480). However, we saw that the Wronskian is nonzero when $x = \pi$, and it is zero when $x = 0$. So, the three functions cannot all be solutions to the differential equation.

8. (20 points) Let

$$A = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix} \ .$$

(a). Compute e^{At}.

As in Example 1 on page 554, we first compute the characteristic polynomial of A. Since A is upper triangular, this is easy: it is $(\lambda - 2)^2$. The matrix has a double eigenvalue of $\lambda = 2$.

We then note that $(A - 2I)^2 = \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. This allows for computing e^{At}:

$$e^{At} = e^{2t} e^{(A-2I)t} = e^{2t} \left(I + t \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix} \right) = e^{2t} \begin{bmatrix} 1 & 3t \\ 0 & 1 \end{bmatrix} \ .$$

(b). Write down the fundamental matrix $X(t)$ for the differential equation $\vec{x}\,' = A\vec{x}$.

It is $e^{At} = \begin{bmatrix} e^{2t} & 3te^{2t} \\ 0 & e^{2t} \end{bmatrix}$.

(c). Write a general solution to $\vec{x}\,' = A\vec{x}$ as a linear combination of vectors.

$$\vec{x}(t) = c_1 \begin{bmatrix} e^{2t} \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 3te^{2t} \\ e^{2t} \end{bmatrix} \ .$$

9. (30 points) (a). For the initial-boundary value problem

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} + 4\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t} \ , & \qquad 0 < x < \pi \ , \quad t > 0 \ , \\ u(0,t) = u(\pi,t) = 0 \ , & \qquad t > 0 \ , \\ u(x,0) = f(x) \ , & \qquad 0 < x < \pi \ , \end{aligned}$$

carry out separation of variables to produce two ordinary differential equations and all corresponding boundary or initial conditions.

Substitute $u(x,t) = X(x)T(t)$ into the original differential equation and divide by $X(x)T(t)$:

$$X''(x)T(t) + 4X'(x)T(t) = X(x)T'(t) \ ;$$
$$\frac{X''(x)}{X(x)} + \frac{4X'(x)}{X(x)} = \frac{T'(t)}{T(t)} \ .$$

Since the left-hand side is independent of t and the right-hand side is independent of x, their common value must equal some constant $-\lambda$.

Looking at the left-hand side first, we have

$$\frac{X''(x)}{X(x)} + \frac{4X'(x)}{X(x)} = -\lambda \ ;$$

therefore

$$X''(x) + 4X'(X) + \lambda X(x) = 0 \ . \tag{1}$$

The boundary conditions $u(0,t) = u(\pi,t) = 0$ give conditions on $X(x)$:

$$X(0) = X(\pi) = 0 \ . \tag{2}$$

For the right-hand side, we have

$$\frac{T'(t)}{T(t)} = -\lambda \ ;$$

therefore

$$T'(t) + \lambda T(t) = 0 \ . \tag{3}$$

The two differential equations are (1)–(2) and (3).

(b). Find an eigenfunction for one of the ordinary differential equations that you found in part (a). [**Correction:** Find an eigenfunction for X.]

The characteristic polynomial of (1) is $r^2 + 4r + \lambda$, which has roots

$$\frac{-4 \pm \sqrt{16 - 4\lambda}}{2} = -2 \pm \sqrt{4 - \lambda} \ .$$

Based on our experience with the heat equation, we look for values of λ that cause the roots to be complex (and not real); i.e., $\lambda > 4$. So, if $\lambda > 4$, then the roots are $-2 \pm i\sqrt{\lambda - 4}$, and the general solution to (1) is

$$X(x) = c_1 e^{-2x} \cos x\sqrt{\lambda - 4} + c_2 e^{-2x} \sin x\sqrt{\lambda - 4}\ .$$

The boundary condition $X(0) = 0$ then gives $c_1 = 0$, and the other boundary condition gives $\sin \pi\sqrt{\lambda - 4} = 0$. This gives that $\sqrt{\lambda - 4}$ must be an integer (necessarily positive).

The choice $\lambda = 5$ gives $\sqrt{\lambda - 4} = 1$, which in turn gives the eigenfunction

$$X(x) = e^{-2x} \sin x\ .$$

10. (20 points) Find a formal solution to the initial-boundary value problem

$$\begin{aligned} \frac{\partial u}{\partial t} &= 3\frac{\partial^2 u}{\partial x^2}\ , \qquad 0 < x < 1\ , \quad t > 0\ , \\ u(0,t) = u(1,t) &= 0\ , \qquad t > 0\ , \\ u(x,0) &= \sum_{n=1}^{\infty} \frac{1}{n^2} \sin n\pi x\ , \qquad 0 < x < 1\ . \end{aligned}$$

This is the heat equation with $\beta = 3$ and $L = 1$. The initial condition gives $c_n = 1/n^2$, so a formal solution (using the given formula) is

$$u(x,t) = \sum_{n=1}^{\infty} \frac{1}{n^2} e^{-3(n\pi)^2 t} \sin n\pi x\ .$$