NAME: Solutions

9 March 2009

Physics 112

Spring 2009

Midterm 1

(50 minutes =50 points)

Try to time yourself. You may use one single sided page of notes.

1) **Thermodynamic identity (7 points)**

a) (5 points) State the thermodynamic identity (1^{st} law of thermodynamics) for

$$G = U - \tau\sigma + PV$$

$$dG = dU - \sigma d\tau - \tau d\sigma + Vdp + pdV$$

$$= \cancel{\tau d\sigma} - \cancel{pdV} + \mu dN - \sigma d\tau - \cancel{\tau d\sigma} + Vdp + \cancel{pdV}$$

$$\boxed{dG = -\sigma d\tau + Vdp + \mu dN}$$

b) (2 points) Identify the natural independent variables (i.e. for which partial derivatives are immediately identifiable in the thermodynamic identity) for the quantity you have chosen for a).

$$\boxed{\tau, \ p, \ N}$$

2) **Energy of a system of harmonic oscillators (10 points).**

We consider a system of N harmonic oscillators of fundamental energy $\hbar\omega$. We have seen in homework that for large N the number of states of energy U is

$$g(U) = \frac{\left(\frac{U}{\hbar\omega} + N\right)^{\frac{U}{\hbar\omega}+N}}{\left(\frac{U}{\hbar\omega}\right)^{\frac{U}{\hbar\omega}} N^N}$$

a. (5 points) Compute the temperature τ as a function of U.

$$\sigma = \log g = \left(\frac{U}{\hbar\omega} + N\right)\log\left(U/\hbar\omega + N\right) - \frac{U}{\hbar\omega}\log\left(\frac{U}{\hbar\omega}\right) - N\log N$$

$$\frac{1}{\tau} = \left(\frac{\partial\sigma}{\partial U}\right)_{N,V} = \frac{1}{\hbar\omega}\log\left(\frac{U}{\hbar\omega} + N\right) + \cancel{\frac{U/\hbar\omega + N}{U/\hbar\omega + N}\frac{1}{\hbar\omega}} - \frac{1}{\hbar\omega}\log\left(\frac{U}{\hbar\omega}\right) - \cancel{\frac{U/\hbar\omega}{U/\hbar\omega}\frac{1}{\hbar\omega}}$$

$$\frac{1}{\tau} = \frac{1}{\hbar\omega}\log\left(\frac{U/\hbar\omega + N}{U/\hbar\omega}\right) = \frac{1}{\hbar\omega}\log\left(1 + \frac{\hbar\omega N}{U}\right)$$

$$\boxed{\tau(U) = \frac{\hbar\omega}{\log\left(1 + \frac{\hbar\omega N}{U}\right)}}$$

b. (3 points) Show that the average energy per oscillator is

$$\frac{U}{N} = \frac{\hbar\omega}{\exp\left(\frac{\hbar\omega}{\tau}\right) - 1}$$

This is the Planck distribution!

From part a: $\tau(u) = \dfrac{\hbar\omega}{\log\left(1+\hbar\omega\frac{N}{u}\right)} \rightsquigarrow \log\left(1+\hbar\omega\frac{N}{u}\right) = \dfrac{\hbar\omega}{\tau}$

$\hbar\omega\frac{N}{u} = e^{\hbar\omega/\tau} - 1$, $\boxed{\dfrac{u}{N} = \dfrac{\hbar\omega}{e^{\hbar\omega/\tau} - 1}}$

3) **Centrifuge (8 points)**

Centrifuges can used to increase the concentration in a solvent of bio-molecules such as DNA. They are often cylinders rotating around their axis at a large angular velocity ω. We will consider molecules of mass m and will assume that the concentrations are small enough for the molecules to be behave as an ideal gas in an incompressible solvent and for the solvent density to be constant with radius (the solvent therefore drops out of the problem). We will assume that the system is in thermal equilibrium at temperature τ.

a) (3 points) Show that the potential energy per particle at radius r is

$$\phi = -\frac{mr^2\omega^2}{2}$$

$F = m\omega^2 r = -\dfrac{\partial\phi}{\partial r}$, $\phi(r) = -\displaystyle\int_0^r m\omega^2 r\, dr = -\frac{1}{2}m\omega^2 r^2$

$\hookrightarrow \phi = -\dfrac{mr^2\omega^2}{2}$

b) (5 points) Use what you know about the chemical potential of an ideal gas to give the variation of concentration n of the bio-molecules with the radius r?

$\mu = \mu_{int} + \mu_{ext} = \tau \log\frac{n}{n_Q} - \frac{1}{2}m\omega^2 r^2 = \text{constant}$

$\mu(r) = \mu(0) \Rightarrow \tau\log\frac{n(r)}{n_Q} - \frac{1}{2}m\omega^2 r^2 = \tau\log\frac{n(0)}{n_Q} - \frac{1}{2}m\omega^2(0)^2$

$\log\frac{n(r)}{n(0)} = \frac{1}{2}m\omega^2 r^2/\tau$

$\boxed{n(r) = n(0)\, e^{\frac{1}{2}m\omega^2 r^2/\tau}}$

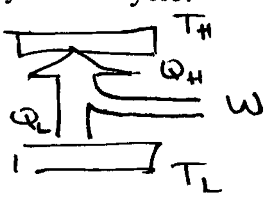
4) **Light bulb problem (10 points)**

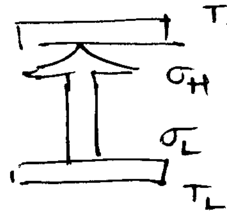
A 100W light bulb is left burning inside a reversible refrigerator that draws 100W.

a) (3 points) Can the refrigerator cool below room temperature?

Yes, if it can reach steady state.

b) (7 points) Justify your answer by drawing the exchanges of energy and entropy and deriving the efficiency of the refrigerator through the conservation of energy and entropy over a cycle.





$\sigma_H = \sigma_L$

$\frac{Q_H}{T_H} = \frac{Q_L}{T_L}$

$W + Q_L = Q_H$

$\eta = \frac{W}{Q_H} = \frac{Q_H - Q_L}{Q_H} = 1 - \frac{Q_L}{Q_H}$

$= 1 - \frac{T_L}{T_H}$

$Q_L = 100$ Watts, $W = 100$ Watts

$Q_H = \frac{W}{\eta} = \frac{100 \text{ Watts}}{\eta}$

$100 \text{ Watts} + 100 \text{ Watts} = \frac{100 \text{ Watts}}{\eta} \Rightarrow \eta = \frac{1}{2} \Rightarrow \boxed{T_H = 2T_L}$

So yes, it can.

5) **Entropy of an ideal gas (15 points)**

This is mostly a conceptual problem, but you will be guided along the way.
We consider an ideal gas system of monoatomic spinless particles of mass M in a volume V at physical temperature τ.

a) (4 points) Using what you know about the density of spatial quantum states per unit phase space volume, show that the number of spatial states for 1 particle between the energy ε and $\varepsilon + d\varepsilon$ (integrated over directions) is

$$2\pi V\left(\frac{2M}{h^2}\right)^{3/2} \varepsilon^{1/2} d\varepsilon$$

phase space density is $1/h^3$, # of states is # in a spherical shell in momentum space of volume $4\pi |\vec{p}|^2 dp$, in a volume V in pos. space.

$\varepsilon = \frac{p^2}{2M}$, $d\varepsilon = \frac{p}{M} dp \rightarrow dp = \sqrt{\frac{M}{2\varepsilon}}\, d\varepsilon$

$= \frac{\sqrt{2M\varepsilon}}{M} dp$

$= \sqrt{\frac{2\varepsilon}{M}}\, dp$

vol: $V \times 4\pi (2M\varepsilon) \sqrt{\frac{M}{2\varepsilon}}\, d\varepsilon$

$= 2\pi V \sqrt{8}\, M^{3/2} \varepsilon^{1/2} d\varepsilon$

$$\boxed{\# \text{ states} = 2\pi V \left(\frac{2M}{h^2}\right)^{3/2} \varepsilon^{1/2} d\varepsilon}$$

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b) (3 points) We have seen in the recent days that the probability for one particle of such system to be in a state of energy ε is proportional to

$$\exp\left(-\frac{\varepsilon}{\tau}\right)$$

Deduce from this that the normalized probability distribution in energy is

$$p(\varepsilon)d\varepsilon = \frac{2}{\sqrt{\pi}\tau^{3/2}}\exp\left(-\frac{\varepsilon}{\tau}\right)\varepsilon^{1/2}d\varepsilon$$

where we have used $\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}dt \quad \Gamma(3/2)=\frac{\sqrt{\pi}}{2}$

$p(\varepsilon) = (\text{normalization}) \times g(\varepsilon) \exp(-\varepsilon/\tau)$

density of states

$= (\text{normalization}) \times \varepsilon^{1/2} e^{-\varepsilon/\tau}$

$\int_0^\infty p(\varepsilon)\,d\varepsilon = (\text{norm}) \int_0^\infty \varepsilon^{1/2} e^{-\varepsilon/\tau}\,d\varepsilon = (\text{norm}) \times \int_0^\infty \tau^{3/2} u^{1/2} e^{-u}\,du$

$= (\text{norm}) \times \tau^{3/2}\,\Gamma(3/2) = 1 \Rightarrow (\text{norm}) = \frac{1}{\tau^{3/2}\Gamma(3/2)} \Rightarrow \boxed{p(\varepsilon) = \frac{2}{\sqrt{\pi}\,\tau^{3/2}} e^{-\varepsilon/\tau} \varepsilon^{1/2}}$

c) (5 points) In order to compute the entropy, we need the probability p_s of individual states of energy ε_s. Comparing the results a and b show that

$$p_s = \frac{1}{V}\left(\frac{h^2}{2\pi M\tau}\right)^{3/2}\exp\left(-\frac{\varepsilon_s}{\tau}\right) \equiv \frac{1}{Vn_Q}\exp\left(-\frac{\varepsilon_s}{\tau}\right)$$

a result we got in class with the partition function.

$p(\varepsilon) = \sum_{\varepsilon_s = \varepsilon} p_s = g(\varepsilon)\, p_s \Rightarrow p_s = \frac{p(\varepsilon)}{g(\varepsilon)} = \frac{\frac{2}{\sqrt{\pi}\,\tau^{3/2}} e^{-\frac{\varepsilon}{\tau}} \varepsilon^{1/2}}{2\pi V \left(\frac{2M}{h^2}\right)^{3/2} \varepsilon^{1/2}}$

$p_s = \frac{1}{\tau^{3/2}} \frac{1}{\pi^{3/2}} \frac{1}{V} \left(\frac{h^2}{2M}\right)^{3/2} e^{-\varepsilon/\tau} = \frac{1}{V}\left(\frac{h^2}{2\pi M\tau}\right)^{3/2} e^{-\varepsilon/\tau} = \frac{1}{Vn_Q} e^{-\varepsilon/\tau}$

d) (3 points) From this result and our **general** definition of entropy, show that the entropy for one particle is

$$\sigma_1 = \log(Vn_Q) + 3/2$$

(this uses the fact that $\Gamma(5/2) = \frac{3}{2}\frac{\sqrt{\pi}}{2}$).

$\sigma = -\sum_s p_s \log p_s \hat{=} -\int d\varepsilon\, p(\varepsilon) \log(p_s(\varepsilon))$

$= -\int_0^\infty d\varepsilon \frac{2}{\sqrt{\pi}} \frac{1}{\tau^{3/2}} e^{-\varepsilon/\tau} \varepsilon^{1/2} \log \frac{1}{Vn_Q} e^{-\varepsilon/\tau}$

$= \frac{2}{\sqrt{\pi}\,\tau^{3/2}} \int_0^\infty \varepsilon^{1/2} e^{-\varepsilon/\tau} \left(\frac{\varepsilon}{\tau} + \log Vn_Q\right) d\varepsilon$

$= \frac{2}{\sqrt{\pi}} \frac{1}{\tau^{5/2}} \underbrace{\int_0^\infty \varepsilon^{3/2} e^{-\varepsilon/\tau}\, d\varepsilon}_{\tau^{5/2}\,\Gamma(5/2)} + \frac{2\log Vn_Q}{\sqrt{\pi}\,\tau^{3/2}} \underbrace{\int_0^\infty \varepsilon^{1/2} e^{-\varepsilon/\tau}\, d\varepsilon}_{\tau^{3/2}\,\Gamma(3/2)}$

$= \frac{2}{\sqrt{\pi}} \frac{3}{2} \frac{\sqrt{\pi}}{2} + \frac{2\log Vn_Q}{\sqrt{\pi}} \frac{\sqrt{\pi}}{2} = \frac{3}{2} + \log Vn_Q \rightarrow \boxed{\sigma_1 = \log Vn_Q + 3/2}$