EECS 70 Discrete Mathematics and Probability Theory

Fall 2014 Anant Sahai

Final exam

Exam location: Back of 1 Pimentel, Second to last digit of SID - 3, 5, or 9

PRINT your student ID: | | | | | | | | |

PRINT AND SIGN your name: ________________, ________________ ____________________

(last) (first) (signature)

PRINT your Unix account login: cs70-____________

PRINT your discussion section and GSI (the one you attend): ______________________________________

Name of the person to your left: ____

Name of the person to your right: ____

Name of someone in front of you: ____

Name of someone behind you: ____

Section 0: Pre-exam questions (3 points)

1. **What are your plans for fun this winter break? (1 pt)**

2. **Describe a time when you overcame an obstacle and succeeded. (2pts)**

Do not turn this page until the proctor tells you to do so.

PRINT your name and student ID: ____

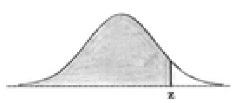
SOME APPROXIMATIONS AND OTHER USEFUL TRICKS THAT MAY OR MAY NOT COME IN HANDY:

$$n! \approx \sqrt{2\pi n}(\frac{n}{e})^n \qquad \binom{n}{k} \le (\frac{ne}{k})^k \qquad \lim_{n\to\infty}(1+\frac{x}{n})^n = e^x$$

When x is small, $\ln(1+x) \approx x$

When x is small, $(1+x)^n \approx 1+nx$

The Golden Rule of 70 (and Engineering generally) applies: if you can't solve the problem in front of you, state and solve a simpler one that captures at least some of its essence. You might get partial credit for doing so, and maybe you'll find yourself on a path to the solution.



Probability Content from -oo to Z

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

Source: `http://www.math.unb.ca/~knight/utility/NormTble.htm`

PRINT your name and student ID: ________________________________

Section 1: Straightforward questions (30 points)

You get one drop: do 5 out of the following 6 questions. Bonus for getting all six perfectly. No partial credit will be given.

3. Prove It

Prove by induction that $5^n - 1$ is divisible by 4 for all integers $n \geq 1$.

PRINT your name and student ID: ______________________________

4. Equate It

Use a combinatorial argument to prove that the following combinatorial identity is true (*i.e.*, provide a story for why the identity should hold).

$$n2^{n-1} = \binom{n}{1} + 2\binom{n}{2} + \cdots + (n-1)\binom{n}{n-1} + n\binom{n}{n}$$

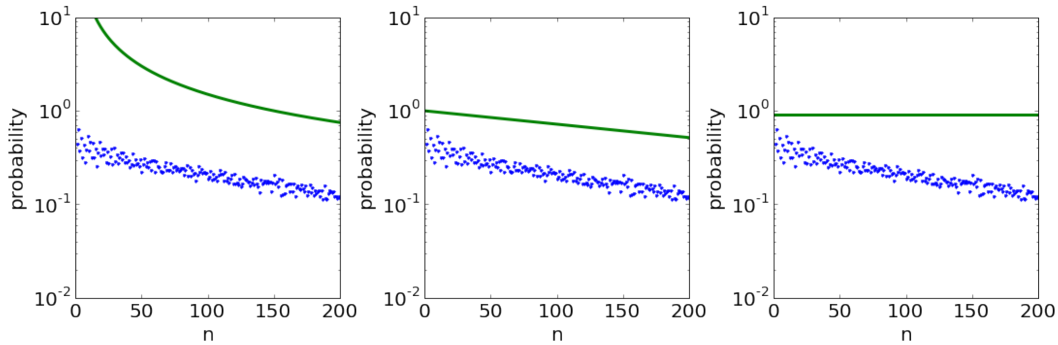
5. Bound It

A random variable X is always strictly larger than -100. You know that $E[X] = -60$. Give the best upper bound you can on $P(X \geq -20)$.

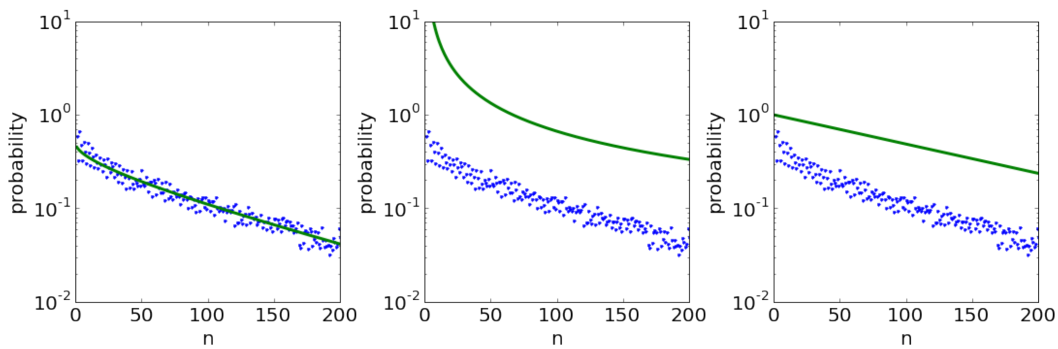
6. Match It

We have n i.i.d. Bernoulli-p random variables. Let A be the average of these random variables. The following plots are on log-scale and show various bounds and approximations for the probability of the average being more than 1.1 times the mean of A as a function of n (depicted in linear scale). The scatter points represent the actual probability of the deviation.

(a) In this set of plots, $p = 0.4$. Label which one corresponds to Markov's Inequality, to Chebyshev's Inequality, and to a Chernoff Bound.



(b) In the next set of plots, $p = 0.6$. Label which one corresponds to Chebyshev's Inequality, to Hoeffding's Inequality, and to the Central Limit Theorem.



7. Expect It

We call a couple a soulmate couple if both of them have each other as the first person on their preference lists. If there are n men and n women, and all of their preference lists are independently uniformly generated over all permutations, what is the expected number of soulmate couples?

8. Provoke It

In a demonstration/protest march, there are 2 kinds of people: civilians and undercover police. You know that 99% of the civilians are peaceful, and 1% advocate violence. You also know that 50% of the undercover police are agent provocateurs (who advocate violence) and 50% of them are peaceful. 2% of the protestors are undercover police. **Given that you see a person in the demonstration advocating violence, what is the probability that this person is undercover police?** (It is fine to leave the answer as a simplified fraction.)

Section 2: Additional straightforward questions (27 points)

You get one drop: do 3 out of the following 4 questions. Bonus for getting all of them perfectly. Very little partial credit will be given.

9. Sections

An EECS class with $3n$ students has three discussion sections, and each student attends exactly one of them by choosing one uniformly at random, independently of the others. **Use Stirling's approximation to estimate an n so that the probability of all three sections having *exactly* the same number of students is about 1.6%.**

It is fine to leave the answer as a simplified formula for n.

(Numerical hints for those who prefer actual numbers: $2\pi \approx 6.28, \sqrt{2\pi} \approx 2.51, \sqrt{3} \approx 1.73.$*)*

10. Two Face

Suppose you have two coins, one that has heads on both sides and another that has tails on both sides.

(a) You pick one of the two coins uniformly at random and you flip that coin 400 times. Approximate the probability of getting more than 220 heads. Your answer should be a number that approximates this probability, accurate to 2 digits after the decimal point.

(b) You pick one of the two coins uniformly at random and flip it. You repeat this process 400 times, each time picking one of the two coins uniformly at random and then flipping it, for a total of 400 flips. Approximate the probability of getting more than 220 heads. Your answer should be a number that approximates this probability, accurate to 2 digits after the decimal point.

(c) Now you pick one of the two coins uniformly at random and flip it four times. You repeat this process 100 times, each time picking one of the two coins uniformly at random and then flipping it four times, for a total of 400 flips. Approximate the probability of getting more than 220 heads. Your answer should be a number that approximates this probability, accurate to 2 digits after the decimal point.

11. Estimating π

One can estimate π by playing darts with a special dartboard shown in figure 1. Assume every time you throw a dart, the dart will always be inside the square. The probability that your dart lands inside the circle is equal to the ratio of the area of the circle to the area of the square, i.e., $\frac{\pi}{4}$. Let X_i be the random variable denoting whether your dart is within the circle after your i-th throw.

How can you estimate π using this experiment? How many times should you throw to ensure your estimation error is within 0.01 with probability at least 95%? (You can just leave the numerical expression of the number of times but not compute the exact value.)

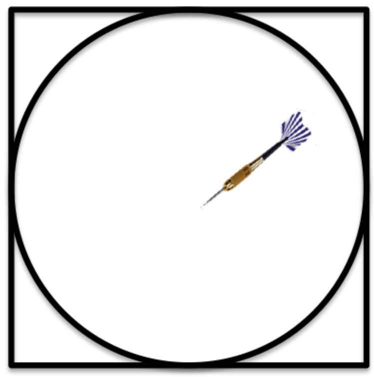


Figure 1: Dartboard.

PRINT your name and student ID: ______________________________

[Extra page. If you want the work on this page to be graded, make sure you tell us on the problem's main page.]

12. Independent Shares

Consider a secret-sharing scheme over $GF(5)$ that divides a secret among 4 people. Assume the secret is uniformly chosen from 0,1,2,3,4. We use the standard polynomial-based secret-sharing scheme so that any 2 people can recover the secret. The linear coefficient in the polynomial is chosen uniformly from 0,1,2,3,4 and independently of the secret.

Show that the shares (i.e. values obtained by evaluating the polynomial at their point) given to person 1 and person 2 are independent.

PRINT your name and student ID: ____

[Extra page. If you want the work on this page to be graded, make sure you tell us on the problem's main page.]

Section 3: True/False (30 points)

You get one drop: do 3 out of the following 4 questions. Bonus for getting all four correct.

For each question in this section, determine whether the given statement is TRUE or FALSE. If TRUE, prove the statement. If FALSE, provide a counterexample or otherwise disprove it.

13. Conditional

If $P(A) > P(B)$, and $P(C|A) > P(C|B)$, then $P(A|C) > P(B|C)$.

Mark one: TRUE or FALSE.

PRINT your name and student ID: ________________________________

14. Epsilon

Consider i.i.d. random variables $\{X_i\}$ with mean μ and variance σ^2. Let $A_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ be the average of n such random variables. Let $g(\varepsilon,\delta)$ be the minimum n so that A_n is within $\pm\varepsilon$ of μ with probability at least $(1-\delta)$. Then for $\delta = 0.1$, we can be certain that $g(\frac{\varepsilon}{2},\delta) \leq 2g(\varepsilon,\delta)$.

Mark one: TRUE or FALSE.

15. Independence

If X,Y are random variables, and $E[XY] = E[X]E[Y]$, then X and Y are independent.

Mark one: TRUE or FALSE.

16. Gotta Get Them All

A person is trying to collect a set of $2n$ cards: n distinct monster cards and n distinct spell cards. When buying a card, she gets a monster card with probability $\frac{2}{3}$ and a spell card with probability $\frac{1}{3}$. Within each category, she will get a card uniformly at random. She keeps buying cards until she has a complete set (owns at least one of each card).

When n is large, the expected number of cards that she buys is less than or equal to $3n(\ln(3n)+1)$.

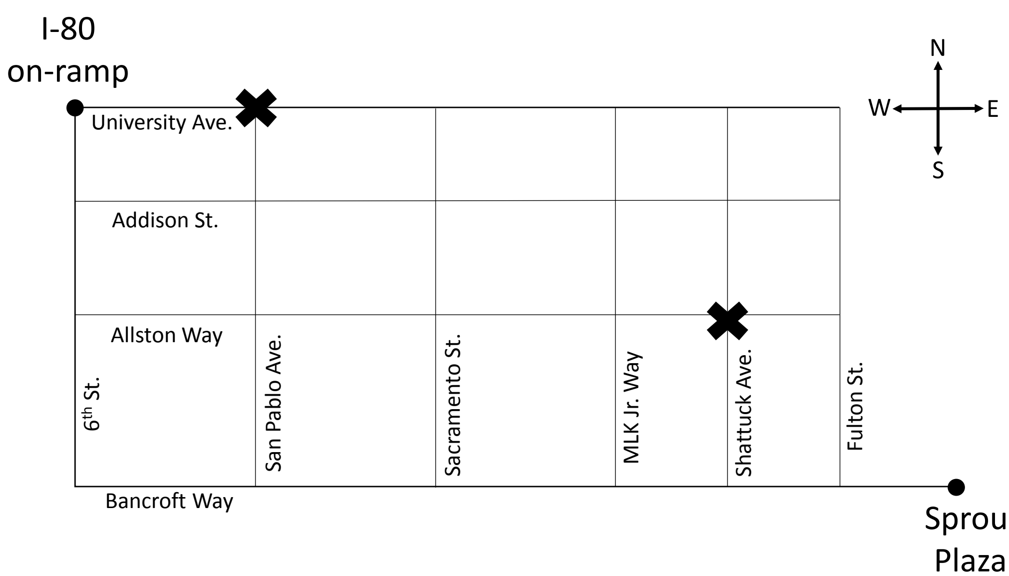
Mark one: TRUE or FALSE.

Section 4: Free-form Problems (73 + 25 points)

17. Peaceful Paths (25 points)

A group of protesters are meeting at Sproul Plaza to march to the I-80 on-ramp. The group doesn't want to backtrack and lose momentum, so they will only move North or West.

For each part, simplify your answer.



(a) (5 points) If the protesters stick to the main paths depicted in the map above, how many possible paths can they take to get from Sproul Plaza (marked by a dot) to the I-80 on-ramp (marked by another dot)?

(b) (5 points) The police have set up blockades (as marked on the map by x's) at the intersection of Allston and Shattuck near the Berkeley BART station and at the intersection of University and San Pablo, just before the on-ramp. How many possible paths can the protesters take such that the police blockades are avoided?

(c) (5 points) Assume the protesters do not know about the blockades, so they are equally likely to choose any path. What is the probability that they are intercepted by a police blockade?

(d) (5 points) In order to increase the chances of some protesters making it to the freeway, the protesters decide to split up into two groups. Each group independently chooses a path to take. As before, either group is equally likely to choose any of the paths. Let X be a random variable denoting the number of groups that make it to the I-80 on-ramp without being intercepted by police. **Write the probability distribution of X.**

(For this part, feel free to use the constants a, b, c to refer to the correct answers to parts a,b,c of this problem.)

(e) (5 points) Now suppose the protesters want to split up into $n > 2$ groups instead of just 2. As before, each of the n groups selects a path independently, and will choose each path with equal probability. **What is the minimum number of groups n needed such that the expected number of groups that reach the I-80 on-ramp without being blocked is at least 1?**

(For this part, feel free to use the constants a, b, c to refer to the correct answers to parts a,b,c of this problem.)

PRINT your name and student ID: ________________________________

[Extra page. If you want the work on this page to be graded, make sure you tell us on the problem's main page.]

18. Couples (15 points)

You are trying to send a message of length 100, and every odd packet drops independently with probability 0.1, and every even packet drops independently with probability 0.5.

(a) (10 points) If you want your message to be successfully received with probability approximately 95%, **about how many additional packets do you have to send using a Reed-Solomon error-correcting code?** Feel free to leave the answer in a form of a simple algebraic equation to be solved.

(b) (5 points) **Is the answer to part (a) more or less than what it would be if the channel dropped packets with probability** 0.3 **at every time?** Why?

19. Miley the Lumberjack (15 points)

For a sequence of numbers $a_1, a_2, \ldots, a_n$, the geometric mean is equal to $(a_1 a_2 \cdots a_n)^{\frac{1}{n}}$. Consider random variables X_i that take on the values

$$X_i = \begin{cases} 2 \text{ with probability } \frac{1}{2}(1-p) \\ 4 \text{ with probability } p \\ 8 \text{ with probability } \frac{1}{2}(1-p) \end{cases}$$

For an i.i.d. sequence of random variables $X_1, X_2, \ldots$ distributed as above with $1 > p > 0$,

$$\lim_{n\to\infty} P\left(\frac{1}{1+\varepsilon} < \frac{(X_1 X_2 \cdots X_n)^{\frac{1}{n}}}{4} < 1+\varepsilon\right) = 0$$

for every $\varepsilon > 0$.

Mark one: TRUE or FALSE.

If TRUE, prove the statement. If FALSE, provide a counterexample or otherwise disprove it.

PRINT your name and student ID: ____________________

[Extra page. If you want the work on this page to be graded, make sure you tell us on the problem's main page.]

PRINT your name and student ID: ________________________________

20. Markov Dice (18 points)

James Bond is playing a game that involves repeatedly rolling a fair standard 6-sided die.

(a) (3 points) What is the expected number of rolls until he gets a 5?

(b) (5 points) What is the expected number of rolls until the last two rolls sum to 7?

(c) (10 points) What is the expected number of rolls until the last two rolls sum to 9?

[Extra page. If you want the work on this page to be graded, make sure you tell us on the problem's main page.]

PRINT your name and student ID: ______________________________

21. (Optional) Digits (25 points)

You choose a number from 0 to 999999 (inclusive) uniformly at random and you sum the digits up. For example if you were to choose 345, the sum of the digits would be 12.

(a) (5 points) What is the probability that the sum is 9?

(b) (10 points) What is the probability that the sum is 19?

(c) (10 points) What is the probability that the sum is 29?

[Extra page. If you want the work on this page to be graded, make sure you tell us on the problem's main page.]

PRINT your name and student ID: ________________________________

[Doodle page! Draw us something if you want or give us suggestions or complaints.]