Lecture 1, Midterm 2, Problem 1:

Speliotopoulos Midterm 2, Fall 2010

$\rho = 3.0 \cdot 10^{-21}$ kg/m^3

$V_{dust} = 0$ (in the reference frame of an outside observer)

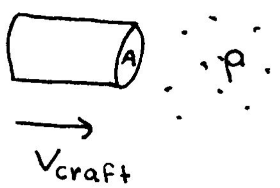
In the spacecraft's frame:

$$\left.\begin{aligned} V^{i}_{dust} &= -V_{craft} \\ V^{f}_{dust} &= +V_{craft} \end{aligned}\right\} \Delta V = 2V_{craft}$$

The momentum imparted to the dust is:

$$\Delta p = m_{dust} \cdot 2V_{craft}$$

The instantaneous change in p depends on the velocity of the craft and on the (instantaneous) rate at which the craft collides with the dust:



$$\frac{dm_{dust}}{dt} = \rho A V_{craft}$$

$$\Rightarrow \frac{dp}{dt} = \frac{dm_{dust}}{dt} \cdot 2V_{craft} = 2\rho A V^2_{craft} = F_{dust}$$

This is the force on the dust. The force on the craft, exerted by the dust, will be equal and opposite, by Newton's 3rd Law:

$$F_{craft} = -F_{dust} = -2\rho A V^2_{craft} = m_{craft} \frac{dV_{craft}}{dt}$$

* Change in notation: "craft" subscripts → "c" subscripts

$$-2\rho A V_c^2 = m_c \frac{dV_c}{dt} = m_c V_c \frac{dV_c}{dx} \quad \left(\frac{dv}{dt} = v\frac{dv}{dx} \text{ ; provided}\right)$$

$$-2\rho A V_c = m_c \frac{dV_c}{dx}$$ use separation of variables:

$$\int_{V_i}^{V_f} \frac{dV_c}{V_c} = \int_0^{x_f} -\frac{2\rho A}{m_c}\, dx$$ (take x = 0 as the point where Voyager enters the galactic disk)

$$\ln\left|\frac{V_f}{V_i}\right| = -\frac{2\rho A}{m_c} x_f, \quad V_f = \frac{1}{2} V_i$$

$$\Rightarrow \ln\left(\frac{1}{2}\right) = -\frac{2\rho A}{m_c} x_f \Rightarrow X_f = \frac{-m_c}{2\rho A} \ln\left(\frac{1}{2}\right) = 8.34 \cdot 10^{22}\ \text{m}$$

1 Problem 2

First the block must overcome static friction to start moving. k has to be large enough to do this.

$$\begin{aligned} kL/2 + kL/2 &> 2\mu mg && (1) \\ k &> 2\mu mg/L && (2) \end{aligned}$$

The next step is to find out where it turns around using energy. Call that distance from the center x.

$$\begin{aligned} E_i &= kL^2/4 \\ E_f &= 1/2kx^2 + 1/2kx^2 + \mu mgL/2 + \mu mgx \\ E_i &= E_f \\ k(L/2)^2 - kx^2 &= \mu mg(L/2 + x) \\ k(L/2 + x)(L/2 - x) &= \mu mg(L/2 + x) \\ k(L/2 - x) &= \mu mg \\ x &= -\mu mg/k + L/2 \end{aligned}$$

Does it turn around or is that momentary stop its last?

$$\begin{aligned} kx + kx = 2kx = 2k(L/2 - \mu mg/k) &= kL - 2\mu mg > 2\mu mg \\ k &> 4\mu mg/L \end{aligned}$$

$L/2 + x$ still isn't far enough so we must require k to be bigger than $4\mu mg/L$. Next stop at position y left of center (Total distance travelled x+y between the two stops)

$$\begin{aligned} E_i &= kx^2 \\ E_f &= 1/2ky^2 + 1/2ky^2 + \mu mgx + \mu mgy \\ E_i &= E_f \\ kx^2 - ky^2 &= \mu mg(y + x) \\ k(y + x)(x - y) &= \mu mg(y + x) \\ k(x - y) &= \mu mg \\ y &= -\mu mg/k + x \\ y &= L/2 - 2\mu mg/k \end{aligned}$$

y is clearly less than x so in order to overcome static friction, k would need to be even bigger than the bounds already placed on it. Therefore if it has travelled far enough already we don't need to bounce again.

Has the block travelled far enough yet?

$$\begin{aligned} d &= L/2 + x + x + y \\ d &= L/2 + L - 2\mu mg/k + L/2 - 2\mu mg/k \\ d &= 2L - 4\mu mg/k \end{aligned}$$

So if $k > 4\mu mg/L$ the block will move from -L/2 to x and then to y covering a distance that is bigger than L. If $k = 4\mu mg/L$ it ends up coming to rest in the center.

Problem 3

Step 1: Energy Conservation (1 point)

$$mgh = \frac{1}{2} m v_0^2$$

$$\boxed{V_0 = \sqrt{2gh}}$$

Step 2: Elastic Collision (11 points)

Generally:

Energy equation: $\frac{1}{2} m_1 V_{01}^2 + \frac{1}{2} m_2 V_{02}^2 = \frac{1}{2} m_1 V_1^2 + \frac{1}{2} m_2 V_2^2$

Momentum equation: $m_1 V_{01} + m_2 V_{02} = m_1 V_1 + m_2 V_2$

Energy $\rightarrow$ $m_2 (V_{02}^2 - V_2^2) = m_1 (V_1^2 - V_{01}^2)$

Momentum $\rightarrow$ $m_2 (V_{02} - V_2) = m_1 (V_1 - V_{01})$ divide

$$V_{02} + V_2 = V_1 + V_{01} \qquad ①$$

Solve along with momentum equation

$$m_1 V_{01} + m_2 V_{02} = m_1 V_1 + m_2 V_2 \qquad ②$$

For this case: $V_{01} = \sqrt{2gh} = V_0$ $\quad V_{02} = 0 \quad m_1 = m \quad m_2 = 7m$

① $V_2 = V_1 + V_0$

② $m V_0 = m V_1 + 7 m V_2$

① → ② $\cancel{m} V_0 = \cancel{m} V_1 + 7 \cancel{m} V_1 + 7 \cancel{m} V_0$

$$-6 V_0 = 8 V_1$$

$$V_1 = -\frac{3}{4} V_0 = -\frac{3}{4}\sqrt{2gh} = -1.66 \text{ m/s}$$

$$V_2 = V_1 + V_0$$

$$V_2 = \frac{V_0}{4} = \frac{1}{4}\sqrt{2gh} = .55 \text{ m/s}$$

Step 3: Going back up hill and turning around (3 points)

There is no friction so the small mass goes up the hill, turns around and comes back with the same speed but opposite direction.

Now $V_1 = \frac{3}{4}V_0$

$$V_1 > V_2$$

Therefore there is a second collision.

Step 4: Elastic Collision #2

$V_{01} = V_1$, $V_{02} = V_2$ from first collision

(5 points)

① $V_2 + V_2' = V_1' + V_1$

② $\cancel{m}V_1 + 7\cancel{m}V_2 = \cancel{m}V_1' + 7\cancel{m}V_2'$

① → $V_2' = V_1' + V_1 - V_2$

Substitute

$$V_1 + 7V_2 = V_1' + 7V_1' + 7V_1 - 7V_2$$

$$8V_1' = 14V_2 - 6V_1$$

$$V_1' = \frac{7}{4}V_2 - \frac{3}{4}V_1 = \frac{7}{16}V_0 - \frac{9}{16}V_0 = -\frac{2}{16}V_0 = -\frac{1}{8}V_0$$

$$V_1' = -\frac{1}{8}V_0$$

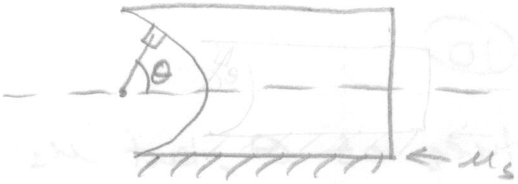
$$V_2' = V_1' + V_1 - V_2 = -\frac{1}{8}V_0 + \frac{3}{4}V_0 - \frac{1}{4}V_0 = \frac{3}{8}V_0$$

After the small mass goes back up the hill and down again

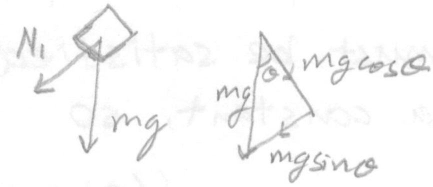
$$\boxed{V_1' = \frac{1}{8}V_0 = \frac{1}{8}\sqrt{2gh} = 0.28 \text{ m/s}} \quad \boxed{V_2' = \frac{3}{8}V_0 = \frac{3}{8}\sqrt{2gh} = 0.83 \text{ m/s}}$$

$V_2' > V_1'$ There are no further collisions.

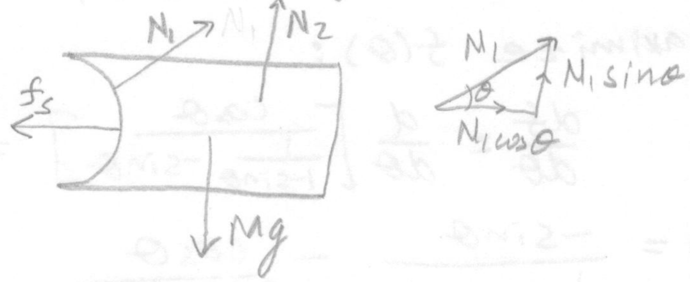
Problem #4



FBD for small block:



FBD for large block:



Big block: $\sum F_x = N_1\cos\theta - f_s = 0$ (1)

$\sum F_y = N_1\sin\theta - Mg + N_2 = 0$ (2)

Small block: $\sum F_R = N_1 + mg\sin\theta = \frac{mv^2}{R}$ (3)

$E_i = mgR + \frac{1}{2}mv_0^2$ (4)

$E_f = mgR\sin\theta + \frac{1}{2}mv^2$ (5)

Combining (4) and (5) $(E_i = E_f)$: $mgR - mgR\sin\theta = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2$

$$\frac{mv_0^2}{R} + 2mg(1-\sin\theta) = \frac{mv^2}{R} \quad (6)$$

Combining (6) and (3): $N_1 + mg\sin\theta = \frac{mv^2}{R} = \frac{mv_0^2}{R} + 2mg(1-\sin\theta)$ (7)

Notice that $N_1 = 0$ at $\theta = \frac{\pi}{2}$, so $mg = \frac{mv_0^2}{R} \Rightarrow v_0^2 = gR$

(7) becomes $N_1 = mg + 2mg(1-\sin\theta) - mg\sin\theta = 3mg(1-\sin\theta)$

From (2), $N_2 = Mg - N_1\sin\theta = Mg - 3mg\sin\theta(1-\sin\theta)$

We need $f_s = N_1\cos\theta$ (equation 1), so $\mu_s N_2 \geq N_1\cos\theta$

$$\mu_s \geq \frac{3mg\cos\theta(1-\sin\theta)}{Mg - 3mg\sin\theta(1-\sin\theta)} \quad (8)$$

$$\mu_s \geq \frac{\cos\theta(1-\sin\theta)}{\frac{M}{3m} - \sin\theta(1-\sin\theta)} \quad (9)$$

Now, $\frac{M}{m} = 3$, so $\frac{M}{3m} = 1$ and (9) becomes:

$$\mu_s \geq \frac{\cos\theta(1-\sin\theta)}{1-\sin\theta(1-\sin\theta)} \quad (10)$$

(10) must be satisfied for all values of θ, but μ_s is a constant, so $\mu_s \geq \max(f(\theta))$ where

$$f(\theta) = \frac{\cos\theta(1-\sin\theta)}{1-\sin\theta(1-\sin\theta)}$$

Let us maximize $f(\theta)$:

$$\frac{df}{d\theta} = \frac{d}{d\theta}\left[\frac{\cos\theta}{\frac{1}{1-\sin\theta} - \sin\theta}\right] = 0$$

$$0 = \frac{-\sin\theta}{\frac{1}{1-\sin\theta} - \sin\theta} - \frac{\cos\theta}{\left(\frac{1}{1-\sin\theta} - \sin\theta\right)^2}\left(-\cos\theta + \frac{\cos\theta}{(1-\sin\theta)^2}\right)$$

$$0 = -\sin\theta\left(\frac{1}{1-\sin\theta} - \sin\theta\right) - \cos\theta\left(-\cos\theta + \frac{\cos\theta}{(1-\sin\theta)^2}\right)$$

$$0 = \frac{-\sin\theta}{1-\sin\theta} + \sin^2\theta + \cos^2\theta - \frac{\cos^2\theta}{(1-\sin\theta)^2}$$

$$0 = -\sin\theta(1-\sin\theta) + (1-\sin\theta)^2 - \cos^2\theta$$

$$0 = -\sin\theta + \sin^2\theta + 1 - 2\sin\theta + \sin^2\theta - \cos^2\theta$$

$$0 = -3\sin\theta + 3\sin^2\theta$$

$$\sin\theta(1-\sin\theta) = 0$$

Either $\sin\theta = 0$ or $\sin\theta = 1$

$\theta = 0$ or $\theta = \frac{\pi}{2}$

$\theta = \frac{\pi}{2}$ clearly is not a maximum for $f(\theta)$.

$$f(0) = \frac{1(1-0)}{1-0(1-0)} = 1$$

so $\max(f(\theta)) = 1$

$$\boxed{\therefore \mu_s \geq 1,\ (\mu_s)_{min} = 1}$$

Problem #5

Let us define V: velocity of

U: velocity of large block in figure B

u': velocity of small block in figure B when it reaches the top of the circle

u_{rel}: velocity of small block in figure B when it reaches the top of the circle, relative to the big block.

There are no external forces acting in the x-direction after the small blocks are launched, and so the center of mass travels at constant velocity:

$$mv_0 = mv + MV \quad (1A) \qquad\qquad mu = mu' + MU \quad (1B)$$

$$V = \frac{m(v_0 - v)}{M} \quad (2A) \qquad\qquad U = \frac{m(u-u')}{M} \quad (2B)$$

at the top, the normal forces satisfy:

$$N_A + mg = \frac{mv_0^2}{R} \quad (3A) \qquad\qquad N_B + mg = \frac{mu_{rel}^2}{R} \quad (3B)$$

$$u_{rel} = u_{m/M} = u' - U \quad (4)$$

In order for the large blocks to just leave the table,

$$N_A = N_B = Mg \quad (5)$$

$$(M+m)g = \frac{mv_0^2}{R} \quad (6a) \qquad\qquad (M+m)g = \frac{mu_{rel}^2}{R} \quad (6b)$$

$$gR = \frac{v_0^2 m}{M+m} \quad (7)$$

Energy is conserved in both figures:

$$E_{i,A} = E_{f,A} \qquad\qquad E_{i,B} = E_{f,B}$$

$$2mgR + \tfrac{1}{2}mv_0^2 = \tfrac{1}{2}mv^2 + \tfrac{1}{2}MV^2 \quad (8a) \qquad \tfrac{1}{2}mu^2 = 2mgR + \tfrac{1}{2}mu'^2 + \tfrac{1}{2}MU^2 \quad (8b)$$

Use (7), (2A), (2B) and simplify:

$$\frac{4m^2v_0^2}{m+M} + mv_0^2 = mv^2 + \frac{Mm^2(v_0-v)^2}{M^2} \qquad mu^2 = \frac{4m^2v_0^2}{m+M} + mu'^2 + \tfrac{1}{2}M(u'-u_{rel})^2$$

Define $x = \frac{m}{M}$, and continue simplifying:

$$\frac{4xv_0^2}{x+1} + v_0^2 = V^2 + xv_0^2 - 2xv_0V + xV^2$$

$$0 = (1+x)V^2 - 2xv_0V + v_0^2\left(x-1-\frac{4x}{x+1}\right)$$

$$V = \frac{2xv_0 \pm \sqrt{4x^2v_0^2 + 4(1+x)\left(\frac{4x}{x+1}+1-x\right)}}{2(1+x)} \quad (10)$$

$$\frac{V}{v_0} = \frac{x \pm \sqrt{x^2+4x+1-x^2}}{1+x}$$

$$\frac{V}{v_0} = \frac{x \pm \sqrt{4x+1}}{1+x}$$

$$\frac{V}{v_0} = \frac{\frac{5}{16} \pm \sqrt{\frac{5}{4}+1}}{\frac{21}{16}}$$

$$\frac{V}{v_0} = \frac{\frac{5}{16} \pm \frac{3}{2}}{\frac{21}{16}}$$

$$\frac{V}{v_0} = \frac{5 \pm 24}{21}$$

We know we want $v - V > 0$ or

$$\frac{v}{v_0} - \frac{V}{v_0} < 0$$

So if we pick $\frac{V}{v_0} = \frac{29}{21}$, then (1A) indicates that $\frac{v}{v_0} < 0$ and so $\frac{v}{v_0} - \frac{V}{v_0} > 0$.

$$\boxed{\therefore \frac{V}{v_0} = -\frac{19}{21}}$$

$$u^2 = \frac{4xv_0^2}{x+1} + u'^2 + \frac{1}{x}(u' - u_{rel})^2$$

$$0 = \frac{1}{x}(u' - u_{rel})^2 + u'^2 - u^2 + \frac{4xv_0^2}{x+1} \quad (9)$$

From (3A) and (3B), we have

$$u_{rel}^2 = v_0^2$$

$u_{rel} = v_0$ since $u_{rel} < 0$ and $v_0 < 0$.

(9) becomes:

$$0 = \frac{1}{x}(u' - v_0)^2 + u'^2 - u^2 + \frac{4xv_0^2}{x+1}$$

From (4) and (2B), we have:

$$u' - x(u - u') = u_{rel} = v_0$$

$$u'(1+x) - xu = v_0$$

$$u' = \frac{v_0 + xu}{1+x}$$

(9) then becomes:

$$0 = \frac{1}{x}\left(\frac{v_0+xu}{1+x} - v_0\right)^2 + \left(\frac{v_0+xu}{1+x}\right)^2 - u^2 + \frac{4xv_0^2}{x+1} \quad (11)$$

$$0 = \frac{1}{x}\left(\frac{x(u-v_0)}{1+x}\right)^2 + \frac{(v_0+xu)^2}{(1+x)^2} - u^2 + \frac{4xv_0^2}{x+1} \quad (12)$$

(12) becomes: $$\frac{x(u^2 - 2v_0u + v_0^2) + v_0^2 + 2xuv_0 + x^2u^2}{(1+x)^2} - u^2 + \frac{4xv_0^2}{x+1} = 0$$

$$\frac{xu^2(1+x) + v_0^2(1+x)}{(1+x)^2} - \frac{u^2 + xu^2}{1+x} + \frac{4xv_0^2}{x+1} = 0$$

$$xu^2 + v_0^2 - u^2 - xu^2 + 4xv_0^2 = 0$$

$$u^2 = v_0^2(1+4x)$$

$u = \pm v_0\sqrt{1+4x} = \pm v_0\frac{3}{2} = -v_0\frac{3}{2}$ because $u > 0$, but $v_0 < 0$

Thus $\frac{V}{u} = \frac{V}{v_0}\Big/\frac{u}{v_0} = \frac{-19}{21}\Big/-\frac{3}{2}$ $\boxed{\therefore \frac{V}{u} = \frac{38}{63}}$