General Instructions:

You must show your work to receive full credit for a correct answer.

Showing your work also facilitates partial credit.

Potentially useful stuff:

Harmonic oscillator Hamiltonian $\widetilde{H}$ in dimensionless variables, and its evals/evecs

$$\widetilde{H} = -\frac{1}{2}\frac{\partial^2}{\partial y^2} + \frac{1}{2}y^2$$

$$\widetilde{H}|\Psi_n\rangle = \frac{E_n}{\hbar\omega}|\Psi_n\rangle = \left(n + \frac{1}{2}\right)|\Psi_n\rangle$$

Harmonic oscillator creation/destruction operators

$$a^+ = \frac{1}{\sqrt{2}}\left(y - \frac{\partial}{\partial y}\right) \qquad a = \frac{1}{\sqrt{2}}\left(y + \frac{\partial}{\partial y}\right)$$

Trigonometric Truths for real ϕ and θ

$$e^{i\phi} = \cos(\phi) + i\sin(\phi)$$

$$\cos(\phi + \theta) = \cos(\phi)\cos(\theta) - \sin(\phi)\sin(\theta)$$

$$\tan(\phi) = \frac{\sin(\phi)}{\cos(\phi)} \qquad \frac{\partial}{\partial\phi}\sin(\phi) = \cos(\phi) \qquad \frac{\partial}{\partial\phi}\cos(\phi) = -\sin(\phi)$$

Problem 1: The Quantum Harmonic Oscillator

(i) (20 points) Show that $y^2 = \frac{1}{2}(a^+ + a)^2$ and $\frac{\partial^2}{\partial y^2} = \frac{1}{2}(a^+ - a)^2$ (cover page has definitions).

(ii) (15 points) Use what you showed in (i) and the commutator $[a, a^+] = 1$ to show $\widetilde{H} = a^+ a + \frac{1}{2}$ (see cover for $\widetilde{H}$)

(iii) (15 points) In an earlier era, much thinking focused on a supposed medium called the aether. If we assume the aether exists and that it exerts a small drag on moving masses, we may expect it to cause transitions between harmonic oscillator eigenstates. As drag is usually velocity-dependent, we might guess the "molecule-aether" interaction term was given by (with $\propto$ meaning proportional to)

$$\hat{V}_{ma} \propto \hat{p}^2 \propto \frac{\partial^2}{\partial y^2}$$

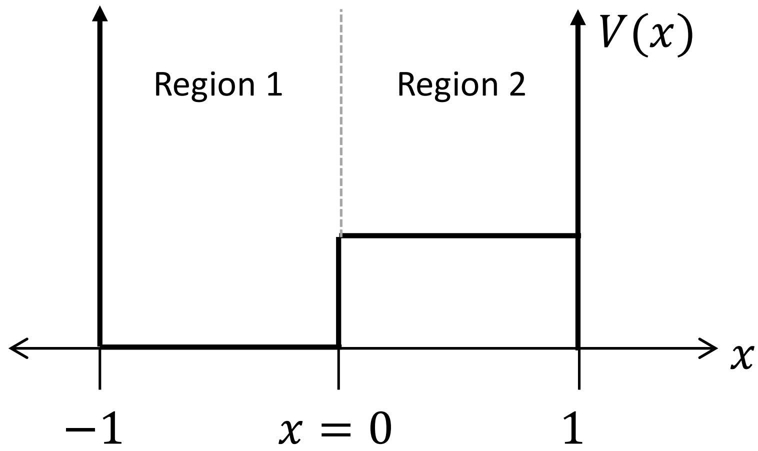
Use what you showed in (i) to find the selection rules for such aether-induced transitions.

(Extra Credit) (15 points) Repeat (iii) above but use $\hat{V}_{ma} = \frac{\partial}{\partial y}$ for the interaction and, instead of creation and destruction operators, use the explicit form of the harmonic oscillator stationary states and the Hermite polynomial relations (both given below, with N_n being a constant for normalization).

$$|\Psi_n\rangle = N_n \, H_n(y) \exp\left(-\frac{1}{2}y^2\right)$$

$$2y \, H_n(y) = H_{n+1}(y) + \frac{\partial H_n(y)}{\partial y} \qquad\qquad \frac{\partial H_n(y)}{\partial y} = 2n \, H_{n-1}(y)$$

Problem 2: Consider the potential below. Also note some useful hyperbolic function definitions.



$$\sinh(x) = \frac{1}{2}(e^{x} - e^{-x})$$

$$\cosh(x) = \frac{1}{2}(e^{x} + e^{-x})$$

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$$

$$\frac{\partial}{\partial x}\sinh(x) = \cosh(x)$$

$$\frac{\partial}{\partial x}\cosh(x) = \sinh(x)$$

$$V(x) = \begin{matrix} 0 & \text{for} & -1 < x < 0 \\ W & \text{for} & 0 < x < 1 \\ \infty & \text{for} & |x| > 1 \end{matrix}$$

We will assume $m = \hbar = 1$ and $E < W$. Let $k = \sqrt{\frac{1}{2}E}$ and $q = \sqrt{\frac{1}{2}(W - E)}$

(i) (10 points) In region 1, where the general solution is $\Psi_1 = c_1\, e^{ikx} + d_1\, e^{-ikx}$, show that choosing

$$c_1 = -\frac{i}{2}e^{ik} \qquad \text{and} \qquad d_1 = \frac{i}{2}e^{-ik}$$

gives us $\Psi_1 = \sin(k(x+1))$ and that this Ψ_1 is well behaved at $x = -1$.

(ii) (10 points) In region 2, where the general solution is $\Psi_2 = c_2\, e^{-qx} + d_2\, e^{qx}$, show that the choice

$$c_2 = -\frac{1}{2}c\, e^{q} \qquad \text{and} \qquad d_2 = \frac{1}{2}c\, e^{-q}$$

gives us $\Psi_2 = c\,\sinh(q(x-1))$ and that this Ψ_2 is well behaved at $x = 1$.

(iii) (20 points) Use the Ψ_1 and Ψ_2 you derived in (i) and (ii) and the conditions they must satisfy at $x = 0$ to show that

$$\frac{\tan(\mathrm{k})}{\mathrm{k}} = -\frac{\tanh(q)}{q}$$

(iv) (10 points) Use the relation from (iii) and the fact that q and k are both positive to show that a tunneling solution is not possible in this system if E is less than $\frac{1}{2}\pi^2$. The following plot may be useful