General Instructions:

You must show your work to receive full credit for a correct answer.

Showing your work also facilitates partial credit.

Potentially useful stuff:

The Schrodinger Equation

$$i\hbar \frac{\partial}{\partial t} |\Psi\rangle = \hat{H} |\Psi\rangle$$

The Heisenberg Uncertainty Principle

$$\Delta A \Delta B \geq \frac{1}{2} \left| \langle [\hat{A}, \hat{B}] \rangle \right|$$

Trigonometric Truths for real ϕ and θ

$$e^{i\phi} = \cos(\phi) + i \sin(\phi)$$

$$\cos(\phi + \theta) = \cos(\phi)\cos(\theta) - \sin(\phi)\sin(\theta)$$

Problem 1: Observables

i) (9 points) Which of the following are linear operators? Show why or why not.

$$\hat{A}\big(f(x)\big) = \cos(x)\, f(x) \qquad \hat{B}\big(f(x)\big) = f(x)\frac{\partial f(x)}{\partial x} \qquad \hat{C}\big(f(x)\big) = -i\hbar\frac{\partial f(x)}{\partial x}$$

ii) (11 points) Can the values of the observables given by $\hat{A}$ and $\hat{C}$ above be known simultaneously? Show why or why not.

Problem 2: Quantum coin flip

You are playing a game with a quantum coin that exists in either of the orthonormal states heads up $|u\rangle$ or heads down $|d\rangle$ or in any superposition of the two. The coin has three observables. First, the heads operator with eigenvectors $|u\rangle$ and $|d\rangle$. Second the Hamiltonian operator $\hat{H}$ for which

$$\hat{H}|u\rangle = |u\rangle + 2|d\rangle \qquad \text{and} \qquad \langle d|\hat{H}|d\rangle = -2$$

Third, you win or lose the game based on the win/loss observable. The eigenvector of this operator that you must observe in order to win is

$$|w\rangle = \frac{1}{\sqrt{2}}|u\rangle + \frac{1}{\sqrt{2}}|d\rangle$$

i) (10 points) Show that $|a\rangle = \frac{2}{\sqrt{5}}|u\rangle + \frac{1}{\sqrt{5}}|d\rangle$ and $|b\rangle = \frac{1}{\sqrt{5}}|u\rangle - \frac{2}{\sqrt{5}}|d\rangle$ are eigenvectors of $\hat{H}$ and find their eigenvalues E_a and E_b.

ii) (10 points) Write $|u\rangle$ as a linear superposition of $|a\rangle$ and $|b\rangle$.

iii) (30 points) If at time $t = 0$ you observe that the coin is heads up, how long should you wait to measure the win/loss observable in order to maximize your chances of winning? You may set $\hbar = 1$.

Problem 3: The Time-Energy Uncertainty Principle (30 points)

The time it takes, on average, for the change of an observable with time to be noticeable is given by

$$\frac{\Delta A}{\left|\frac{\partial\langle\hat{A}\rangle}{\partial t}\right|}$$

The time-energy uncertainty principle states that the time it takes to notice a change depends on how precisely we know the system's energy. Assuming $\hat{A}$ is time independent such that $\frac{\partial\langle\hat{A}\rangle}{\partial t} = \frac{\partial\langle\Psi|}{\partial t}\hat{A}|\Psi\rangle + \langle\Psi|\hat{A}\frac{\partial|\Psi\rangle}{\partial t}$, derive the time-energy uncertainty principle, which states that

$$\frac{\Delta A}{\left|\frac{\partial\langle\hat{A}\rangle}{\partial t}\right|} \geq \frac{\hbar}{2\,\Delta H}$$